Amirkabir University of Technology
(Tehran Polytechnic)

Amirkabir International Jounrnal of Science & Research
Modeling, Identification, Simulation and Control
(AIJ-MISC)
Vol. 48, No. 2, Fall 2016, pp. 111-121

Time-Invariant State Feedback Control Laws for a Special Form of Underactuated Nonlinear Systems Using Linear State Bisection

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(Received 22 February 2016, Accepted 3 July 2016)

Abstract

Linear state bisection is introduced as a new method to find time-invariant state feedback control laws for a special form of underactuated nonlinear systems. The specialty of the systems considered is that every unactuated state should be coupled with at least two directly actuated states. The basic idea is based on bisecting actuated states and using linear combinations with adjustable parameters to stabilize the unactuated states. These linear combinations make the underactuated system virtually fully actuated, making it suitable to be stabilized with well-known nonlinear control methods, like feedback linearization. In addition to its simplicity, one of the main contributions of this method is that it can be applied to the systems with more than one unactuated state. Three underactuated systems are considered: An asymmetric rigid body, a planar rigid body with an unactuated internal degree of freedom and a system with two degrees of underactuation. It is shown through simulations that the proposed control laws can be effectively used to stabilize the special form of underactuated systems considered.

Keywords:

Underactuation, Feedback Linearization, State bisection

Please cite this article using:
Moradi, R., Alikhani, A, and Fathi-Jegarkandi, M., 2016. "Time-Invariant State Feedback Control Laws for a Special Form of Underactuated Nonlinear Systems Using Linear State Bisection". *Amirkabir International Journal of Modeling, Identification, Simulation and Control*, 48(2), pp. 111–121.
DOI: 10.22060/miscj.2016.836
URL: http://miscj.aut.ac.ir/article_836.html
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1- INTRODUCTION

A mechanical system is said to be underactuated when the number of independent control inputs is less than the number of degrees of freedom to be controlled. An extensive amount of studies have been published in the literature to reduce the problems accompanying these systems [1-4], just to mention a few.

Due to their unique dynamic features, developing control laws for a general class of underactuated mechanical systems is a very challenging issue. Therefore, among the works which have considered the control of underactuated systems, most of them have performed a case by case analysis: the ball and beam [5], vertical takeoff and landing [6], cranes [7] and inertial wheel pendulums [8], surface vessels [9], underactuated spacecrafts [10] are only a small representative of the numerous papers published in references, regarding underactuated mechanical systems control. Therefore, establishing control laws for a rather general class of underactuated mechanical systems remains still an open complex problem.

One of the well-established facts about several underactuated systems is that they do not satisfy Brockett's necessary conditions [11] for smooth time-invariant feedback stabilization. This is mainly contributed to the fact that the linearized model of these systems around equilibrium points is uncontrollable, especially in the absence of gravitational terms [1]. Due to this fact, [12] has considered discontinuous time-invariant feedback while [13] have developed time-varying smooth feedback controllers for these systems.

In this paper, a new method called *linear state bisection* (LSB) is proposed to find time-invariant state feedback controls to make the origin asymptotically stable for a special form of underactuated nonlinear systems. The specialty comes from the fact that the considered systems consist of more than two states and it is assumed that the unactuated state is directly affected by at least two actuated states (similar to the class of systems considered by [12]). The main concept of this approach is to linearly bisect a state that has the most influence on the unactuated state. Then, one of these parts is used as a virtual input to control the unactuated state and the other part is controlled through the independent input such that the original assistant state converges into the origin.

In addition to its simplicity, one of the main contributions of this paper compared to other works, e.g. ([14] or [12]) is that the proposed method can be applied to systems with more than one unactuated state.

In order to implement the proposed method, three mechanical systems are considered: An asymmetric underactuated rigid body, a planar rigid body with an unactuated internal degree of freedom and a system with two degrees of underactuation. All of the considered systems are nonlinear (especially the second system which is indeed highly nonlinear) and do not satisfy Brockett's necessary condition. The quality of responses shows that the proposed method can be effectively used to make the origin asymptotically stable for the dynamical systems considered.

The paper is organized as follows: section 2 is devoted to the development of time-invariant state-feedback control laws for the special form of underactuated nonlinear systems considered. All necessary assumptions are given and a theorem is provided to prove that the proposed control laws can make the origin asymptotically stable. In section 3, the obtained results will be used to make the origin asymptotically stable for an asymmetric underactuated rigid body, a planar rigid body with an unactuated internal degree of freedom and a nonlinear system that consists of two unactuated degrees of freedom. The obtained results demonstrate that in spite of being simple, the obtained results can be effectively utilized to stabilize the underactuated systems considered. Finally, section 4 ends the paper with a conclusion and a look at future challenges.

2- LINEAR STATE BISECTION

Consider the following nonlinear system affine in control:

$$\dot{\bar{x}} = f(\bar{x}) + g(\bar{x})\bar{u} \tag{1}$$

where f and g are sufficiently smooth on a compact set of $X \in R^{n+m}$. $\bar{x} \in R^{n+m}$ and $\bar{u} \in R^{n}$ are the state and control vectors, respectively. It is assumed that the n number of states is actuated (or directly affected by the inputs) and m number of them is unactuated although it is assumed that the vector $\bar{x}$ is fully observable.

The goal of the proposed controller is to make the origin a globally asymptotically stable equilibrium for system (1).

Divide this system into two subsystems, actuated and unactuated:

$$\begin{aligned}\dot{\bar{x}}_a &= f_a(\bar{x}_a,\bar{x}_u) + g_a(\bar{x}_a,\bar{x}_u)\bar{u}\\ \dot{\bar{x}}_u &= f_u(\bar{x}_a,\bar{x}_u)\end{aligned} \tag{2}$$

$\bar{x}_a = \left[x_{a_1}, x_{a_2}, \ldots, x_{a_n}\right]$ and $\bar{x}_u = \left[x_{u_1}, x_{u_2}, \ldots, x_{u_m}\right]$ denote respectively, the actuated and unactuated parts of the state vector.

Assumption 1: Each unactuated state is coupled with at least two directly actuated states and therefore, the total number of states is more than two.

A. Systems with One Unactuated State

For case with $m = 1$, expanding (2) in terms of its components will lead to the following equations:

$$\begin{aligned}\dot{x}_{a_1} &= f_{a_1}(\bar{x}_a, x_u) + g_{a_1}(\bar{x}_a, x_u)u_1\\ \dot{x}_{a_2} &= f_{a_2}(\bar{x}_a, x_u) + g_{a_2}(\bar{x}_a, x_u)u_2\\ &\vdots\\ \dot{x}_{a_n} &= f_{a_n}(\bar{x}_a, x_u) + g_{a_n}(\bar{x}_a, x_u)u_n\\ \dot{x}_u &= f_u(\bar{x}_a, x_u)\end{aligned} \tag{3}$$

In order to stabilize the unactuated state (x_u), one of the actuated states e.g. x_{a_n} (assistant state variable) is divided into two parts:

$$x_{a_n} = a x_{a_{n1}} + b x_{a_{n2}}, \quad a, b \in R \tag{4}$$

where a and b are real-valued numbers. It will be shown that the obtained control laws are independent of these two values. In other words, any linear combination of $x_{a_{n1}}$ and $x_{a_{n2}}$ can be used to stabilize the unactuated state.

Eliminating unimportant functionalities and substituting (4) in (3) will lead to (5):

$$\begin{aligned}\dot{x}_{a_1} &= f_{a_1} + g_{a_1}u_1\\ \dot{x}_{a_2} &= f_{a_2} + g_{a_2}u_2\\ &\vdots\\ a\dot{x}_{a_{n1}} &= f_{a_n} + g_{a_n}u_n - b\dot{x}_{a_{n2}}\\ \dot{x}_u &= f_u(x_{a_1}, \ldots, x_{a_{n-1}}, a x_{a_{n1}} + b x_{a_{n2}}, x_u)\end{aligned} \tag{5}$$

The following definition is introduced:

$$\dot{x}_{a_{n2}} = w \tag{6}$$

where w is a new scalar variable and will be used to stabilize the unactuated state variable, x_u. The goal is now to determine $x_{a_{n2}}$ such that x_u approaches the origin. This $x_{a_{n2}}$ is called $x_{a_{n2,des}}$.

Exponential convergence of x_u means that it should satisfy $\dot{x}_u = -k_u x_u$. Substituting the latter in the last equation of (5) will result in the following equation:

$$-k_u x_u = f_u\left(x_{a_1}, \ldots, x_{a_{n-1}}, a x_{a_{n1}} + b x_{a_{n2,des}}, x_u\right) \tag{7}$$

It is assumed that a unique solution exists for $x_{a_{n2,des}}$ in the following general explicit form:

$$x_{a_{n2,des}} = \frac{-a}{b} x_{a_{n1}} + \frac{1}{b}\phi\left(k_u, x_u, x_{a_1}, \ldots, x_{a_{n-1}}\right) \tag{8}$$

where ϕ is a new function that is obtained when (7) is solved for $x_{a_{n2,des}}$.

Taking the limit of (8) as time approaches infinity will result in (9):

$$\lim_{t\to\infty} x_{a_{n2,des}} = \frac{-a}{b}\lim_{t\to\infty} x_{a_{n1}} + \frac{1}{b}\lim_{t\to\infty}\phi \tag{9}$$

Meanwhile, doing the same for (4) will lead to:

$$\lim_{t\to\infty} x_{a_n} = a\lim_{t\to\infty} x_{a_{n1}} + b\lim_{t\to\infty} x_{a_{n2}} \tag{10}$$

Since:

$$\lim_{t\to\infty} x_{a_{n2,des}} = \lim_{t\to\infty} x_{a_{n2}} \tag{11}$$

Inserting (9) in (10) together with (11) will result in (12):

$$\lim_{t\to\infty} x_{a_n} = \lim_{t\to\infty}\phi \tag{12}$$

According to (12), the value of x_{a_n} tends to the value of ϕ as time approaches infinity. Therefore, in order for the steady state value of x_{a_n} to remain bounded, $\lim_{t\to\infty}\phi$ should be bounded as well.

Fortunately, as will be seen in the simulation section (where a model-based analysis will be made), the steady state behavior of ϕ can be adjusted through controller parameters for the considered systems.

In order to transform (5) into a virtually fully actuated form, the following new variable is introduced:

$$z = c x_{a_{n2}} + d x_{a_{n2,des}}, \quad c, d \in R \tag{13}$$

where c and d are real-valued numbers.

It will be shown that unlike a and b, the control inputs will be dependent on c and d and therefore, their values will have very important effects on the performance of the proposed controller.

Remark 1: An important point is that if $c = -d$ and $z = 0$, $x_{a_{n2}}$ will be equal to $x_{a_{n2,des}}$ which is the ideal case. This is equivalent to saying that stabilizing z is equivalent to the stabilization of x_u (remember the previously made statement that if $x_{a_{n2}}$ approaches $x_{a_{n2,des}}$, x_u will approach the origin). According to this point, the equation for $\dot{x}_u$ will be replaced by $\dot{z}$.

Differentiating (13) with respect to time, together with (6) will result in (14):

$$\dot{z} = c\dot{x}_{a_{n2}} + d\dot{x}_{a_{n2,des}} = cw + d\dot{x}_{a_{n2,des}} \tag{14}$$

Substituting (14) for the last equation of (5) will produce (15):

$$\begin{aligned} \dot{x}_{a_1} &= f_{a_1} + g_{a_1}u_1 \\ \dot{x}_{a_2} &= f_{a_2} + g_{a_2}u_2 \\ &\vdots \\ \dot{x}_{a_{n1}} &= \frac{1}{a}\left(f_{a_n} + g_{a_n}u_n - bw\right) \\ \dot{z} &= cw + d\dot{x}_{a_{n2,des}} \end{aligned} \tag{15}$$

The time derivative of $x_{a_{n2,des}}$ can be obtained by partial differentiation of (8):

$$\dot{x}_{a_{n2,des}} = \frac{1}{b}\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\dot{x}_{a_i} - \frac{a}{b}\dot{x}_{a_{n1}} \tag{16}$$

In writing (16), it is assumed that $\frac{\partial\phi}{\partial x_u}\dot{x}_u \approx 0$ in comparison to $\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\dot{x}_{a_i}$. The reason for this assumption as will be seen in the next paragraph is to rewrite in terms of the original states.

Using (4), (6), (16) and performing some mathematical operations (Appendix A), $\dot{z}$ will be simplified to:

$$\dot{z} = (c+d)w + \frac{d}{b}\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\dot{x}_{a_i} - \frac{d}{b}\dot{x}_{a_n} \tag{17}$$

Therefore, the entire set of (15) in a virtually fully actuated form will be given according to (18):

$$\begin{aligned} \dot{x}_{a_1} &= f_{a_1} + g_{a_1}u_1 \\ \dot{x}_{a_2} &= f_{a_2} + g_{a_2}u_2 \\ &\vdots \\ \dot{x}_{a_{n1}} &= \frac{1}{a}\left(f_{a_n} + g_{a_n}u_n - bw\right) \\ \dot{z} &= (c+d)w + \frac{d}{b}\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\dot{x}_{a_i} - \frac{d}{b}\dot{x}_{a_n} \end{aligned} \tag{18}$$

Since w has been introduced as a virtual control input and will be entrusted to lead z towards zero, a very important fact can be deduced from the last equation of (18):

$$c \neq -d \tag{19}$$

However, this is in contradiction with the previously made conclusion i.e. for $z=0$, it is required for c to be equal to $-d$. In order to alleviate this problem, it will be assumed that $c \approx -d$.

Using feedback linearization and demanding $\dot{z} = -k_z z$, w will be obtained as follows:

$$w = \frac{1}{c+d}\left(-k_z z - \frac{d}{b}\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\dot{x}_{a_i} + \frac{d}{b}\dot{x}_{a_n}\right), \quad c \neq -d \tag{20}$$

Now, inserting w in the expression for $\dot{x}_{a_{n1}}$ in (18) and taking into account (13) and the following assumption and lemma:

Assumption 2: $k_{a_{n1}} = \frac{c}{c+d}k_z, k_z = k_{a_n}$

Lemma 1: $\lim_{t\to\infty} x_{a_{n2,des}} = 0$ (Proof in Appendix B)

where $k_{a_{n1}}$, k_z and k_{a_n} are the exponential convergence rate of decay for $x_{a_{n1}}$, z and x_{a_n} respectively, the following result will be obtained for u_n (Proof in appendix C):

$$u_n = \frac{1}{g_n}\left(-k_{a_n}x_{a_n} - f_{a_n} + \frac{d}{c+d}\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\left(k_{a_i}x_{a_i}\right)\right) \tag{21}$$

under the assumption that x_{a_n} is converging into the origin with an approximately exponential rate of convergence:

$$\dot{x}_{a_n} \approx -k_{a_n}x_{a_n} \tag{22}$$

Remark 2: It should be noted that the approximate exponential convergence of x_{a_n} is an ideal assumption because of its role to stabilize x_u. However, as will be seen, this assumption is necessary to derive (21) and consequently, to make the origin asymptotically stable for (3).

The next theorem gives the main results of the paper:

Theorem 1

The following time-invariant state-feedback control laws will make the origin asymptotically stable for system (3):

$$\begin{aligned} u_1 &= \frac{1}{g_{a_1}}\left(-k_{a_1}x_{a_1} - f_{a_1}\right) \\ u_2 &= \frac{1}{g_{a_2}}\left(-k_{a_2}x_{a_2} - f_{a_2}\right) \\ &\vdots \\ u_{n-1} &= \frac{1}{g_{a_{n-1}}}\left(-k_{a_{n-1}}x_{a_{n-1}} - f_{a_{n-1}}\right) \\ u_n &= \frac{1}{g_{a_n}}\left(-k_{a_n}x_{a_n} - f_{a_n} + \frac{d}{c+d}\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\left(k_{a_i}x_{a_i}\right)\right) \end{aligned} \tag{23}$$

Proof: In order to show that the proposed time-invariant state-feedback control laws (23) make the origin globally asymptotically stable for (3), $u_1, u_2, \ldots, u_n$ (23) and w (20) are substituted in (18) considering *assumption 2* and *remark 2*. The result will be:

$$\begin{aligned}
\dot{x}_{a_1} &= -k_{a_1} x_{a_1} \\
\dot{x}_{a_2} &= -k_{a_2} x_{a_2} \\
&\vdots \\
\dot{x}_{a_{n_1}} &= -k_z \frac{c}{c+d} x_{a_{n_1}} \\
\dot{z} &= -k_z z
\end{aligned} \tag{24}$$

If $\frac{c}{c+d}$ is assumed to be positive, (24) will be in the form of $\dot{\bar{x}} = -K\bar{x}$ with K being positive definite. Therefore, all the eigenvalues will be strictly on the left-hand side of the complex plane and the equilibrium point of (24) will be globally exponentially stable. A more detailed proof is presented in Appendix D.

The next lemma will complete the theorem:

Lemma 2: x_{a_n} and x_u converge to the origin.

Proof: According to the last equation of (24) and (13):

$$\lim_{t\to\infty} z = \lim_{t\to\infty}\left(c x_{a_{n2}} + d x_{a_{n2,des}}\right) = 0 \tag{25}$$

and because it was assumed that $c \approx -d$, $x_{a_{n2}}$ will approach $x_{a_{n2,des}}$ which is according to the definition of $x_{a_{n2,des}}$, equivalent to the exponential convergence of x_u to the origin.

On the other hand, according to *lemma 1*, $\lim_{t\to\infty} x_{a_{n2,des}} = 0$. Therefore, $x_{a_{n2}}$ will converge into zero and since $x_{a_{n1}}$ converges into the origin, according to (4), x_{a_n} will be asymptotically stabilized. It is obvious that the states $x_{a_1} .. x_{a_{n-1}}$ will converge into the origin exponentially. Consequently, the origin will be an asymptotically stable equilibrium point for (3).

This completes the proof of the theorem.

From (21), the control input u_n consists of two parts:

$$\begin{aligned}
u_{n_1} &= \frac{-1}{g_{a_n}}\left(k_{a_n} x_{a_n} + f_{a_n}\right) \\
u_{n_2} &= \frac{1}{g_{a_n}} \frac{d}{c+d} \sum_{i=1}^{n-1} \frac{\partial \phi}{\partial x_{a_i}}\left(k_{a_i} x_{a_i}\right)
\end{aligned} \tag{26}$$

u_{n_1} and u_{n_2} are used to stabilize x_{a_n} and x_u respectively. Because of its role in controlling x_u, u_{n_2} will be called the *assistant input*.

Remark 3: It can be easily concluded that if f_u is not a function of at least one directly actuated state (in addition to the assistant state), or equivalently, ϕ a function of at least one directly actuated state, u_{n_2} will be vanished and therefore the unactuated state will not be asymptotically stabilized.

Remark 4: Although developed for systems with one unactuated state, the procedure to extend the LSB method to underactuated systems with more than one unactuated state is straightforward. This procedure will be followed in the next section.

B. Systems with more than One Unactuated State

Consider (2) and assume that the number of unactuated states is more than one, i.e. $m > 1$. Therefore (3) can be written as:

$$\begin{aligned}
\dot{x}_{a_1} &= f_{a_1}(\bar{x}_a, \bar{x}_u) + g_{a_1}(\bar{x}_a, \bar{x}_u) u_1 \\
\dot{x}_{a_2} &= f_{a_2}(\bar{x}_a, \bar{x}_u) + g_{a_2}(\bar{x}_a, \bar{x}_u) u_2 \\
&\vdots \\
\dot{x}_{a_n} &= f_{a_n}(\bar{x}_a, \bar{x}_u) + g_{a_n}(\bar{x}_a, \bar{x}_u) u_n \\
\dot{x}_{u_1} &= f_{u_1}(\bar{x}_a, \bar{x}_u) \\
&\vdots \\
\dot{x}_{u_m} &= f_{u_m}(\bar{x}_a, \bar{x}_u)
\end{aligned} \tag{27}$$

If i and j refer to the i-th actuated and j-th unactuated states respectively:

$$1 \le i \le n,\ 1 \le j \le m \tag{28}$$

Taking into account the concept of LSB, in addition to the necessity of the presence of an actuated (assistant) state for every unactuated state, there should be a coupling between other directly actuated (other than assistant states) and unactuated states, exactly similar to the case with $m = 1$.

Assume there is a correspondence between $[\phi_1 .. \phi_m]$ and $\left[f_{u_1} .. f_{u_m}\right]$ and u_i is the i-th control input dedicated to control the j-th unactuated state:

$$\begin{aligned}
u_{i_1} &= \frac{-1}{g_i}\left(k_{a_i} x_{a_i} + f_{a_i}\right) \\
u_{i_2} &= \frac{1}{g_i} \frac{d_j}{c_j + d_j} \sum_{l=1}^{n-m} \frac{\partial \phi_j}{\partial x_{a_l}}\left(k_l x_{a_l}\right)
\end{aligned} \tag{29}$$

(29) is the general form of (26) for the case $m > 1$.

3- Case Studies

In this section, three case studies are considered: an asymmetric rigid body with two control inputs, a planar rigid body with an unactuated internal degree of freedom and a system with two degrees of underactuation.

A. Angular Velocity Stabilization of an Underactuated Rigid Body

Rigid body angular velocity equations are described by the well-known Euler's relations [15]:

$$\begin{aligned}\dot{p} &= \alpha_1 qr + u_1\\ \dot{q} &= \alpha_2 pr + u_2\\ \dot{r} &= \alpha_3 pq + u_3\end{aligned} \tag{30}$$

p, q and r and are the angular velocities of the rigid body with respect to an inertial reference frame and expressed in body coordinates. u_1, u_2 and u_3 are the normalized control inputs and α_i's are a fraction of the moments of inertia and are assumed to be constant. Their values are obtained from:

$$\begin{aligned}\alpha_1 &= \frac{J_y - J_z}{J_x}\\ \alpha_2 &= \frac{J_z - J_x}{J_y}\\ \alpha_3 &= \frac{J_x - J_y}{J_z}\end{aligned} \tag{31}$$

J_x, J_y and J_z are the principal moments of inertia of the rigid body along the body axis. The relation between control torques and inputs are given by the following equations:

$$\begin{aligned}u_1 &= \frac{M_x}{J_x}\\ u_2 &= \frac{M_y}{J_y}\\ u_3 &= \frac{M_z}{J_z}\end{aligned} \tag{32}$$

M_x, M_y and M_z are the three control moments acting on the spacecraft.

Without losing generality, it is assumed that $u_3 = 0$. On the other hand, it is assumed that the rigid body under study is asymmetric. In other words: $\alpha_i \neq 0 \ \forall i = 1,2,3$.

Taking into account these two assumptions, (30) will be simplified to (33):

$$\begin{aligned}\dot{p} &= \alpha_1 qr + u_1\\ \dot{q} &= \alpha_2 pr + u_2\\ \dot{r} &= \alpha_3 pq\end{aligned} \tag{33}$$

In order to continue, the following changes of variables are introduced:

$$\begin{aligned}x_{a_1} &= p\\ x_{a_2} &= q\\ x_u &= r\end{aligned} \tag{34}$$

Therefore, (33) will be rewritten as:

$$\begin{aligned}\dot{x}_{a_1} &= \alpha_1 x_{a_2} x_u + u_1\\ \dot{x}_{a_2} &= \alpha_2 x_{a_1} x_u + u_2\\ \dot{x}_u &= \alpha_3 x_{a_1} x_{a_2}\end{aligned} \tag{35}$$

(35) is a special form of (3). On the other hand, both x_{a_1} and x_{a_2} have the same effect on x_u and therefore x_{a_2} is selected as the assistant state variable. Since $f_u = \alpha_3 x_{a_1} x_{a_2}$ is a function of x_{a_1}, LSB can be used to make the origin asymptotically stable for (35). It should be noted that in this example $n = 2$ and $m = 1$.

The following state bisection is introduced:

$$x_{a_2} = a x_{a_{21}} + b x_{a_{22}} \tag{36}$$

Inserting (36) into the last equation of (35), $x_{a_{22,des}}$ will be given by (37):

$$x_{a_{22,des}} = \frac{-a}{b} x_{a_{21}} - \frac{1}{b}\frac{k_u x_u}{\alpha_3 x_{a_1}} \tag{37}$$

Therefore, according to (8), ϕ will be:

$$\phi = \frac{-k_u x_u}{\alpha_3 x_{a_1}} \tag{38}$$

According to (38), if the rate of convergence for x_u towards the origin is larger than x_{a_1}, the numerator approaches the origin faster than the denominator and the steady-state value of ϕ tends to zero as time goes to infinity.

Taking the derivative of (38) with respect to x_{a_1} will result in (39):

$$\frac{\partial \phi}{\partial x_{a_1}} = \frac{k_u x_u}{\alpha_3 x_{a_1}^2} \tag{39}$$

Finally, according to (26), u_2 will be:

$$u_2 = -k_{a_2} x_{a_2} - \alpha_2 x_{a_1} x_u + \frac{d}{c+d}\frac{k_{a_1} k_u x_u}{\alpha_3 x_{a_1}} \tag{40}$$

and according to the first equation of (23), u_1 will be:

$$u_1 = -k_{a_1} x_{a_1} - \alpha_1 x_{a_2} x_u \tag{41}$$

In terms of the original variables, u_1 and u_2 will be:

$$\begin{aligned}u_1 &= -k_p p - \alpha_1 qr\\ u_2 &= -k_q q - \alpha_2 pr + \frac{d}{c+d}\frac{k_p k_r r}{\alpha_3 p}\end{aligned} \tag{42}$$

These control inputs are similar to those obtained by [16].

According to (42), the larger the roll rate of the rigid body (p), the less pitch rate (q) will be required to control yaw rate (r). Physically speaking, this is the well-known *gyroscopic effect* that occurs in rolling objects [15].

According to theorem 1, (42) will make the origin asymptotically stable for the nonlinear system (33).

The initial conditions are chosen as $p_0 = 8, q_0 = -6, r_0 = 7$ deg/sec. The control coefficients are selected to be $k_p = 0.05, k_q = 0.1, k_r = 0.1$. On the other hand, according to (23), $\frac{c}{c+d}$ should be positive. Therefore c and d are selected as 1, -0.9, respectively.

The asymmetric rigid body is assumed to be an underactuated rigid spacecraft whose physical parameters are selected as [17]:

$J_x = 449.5, J_y = 264.6, J_z = 312.5$ kg.m^2

The controller capability to steer the states towards the origin is shown in Figs. 1 and 2.

It is shown that the underactuated spacecraft is asymptotically stabilized using the proposed control laws. The controller parameters were selected such that the convergence rate of p is smaller than the other two states.

Although a singularity exists at p=0, due to the smoothness of the control inputs, no boundary layer has been defined in this case.

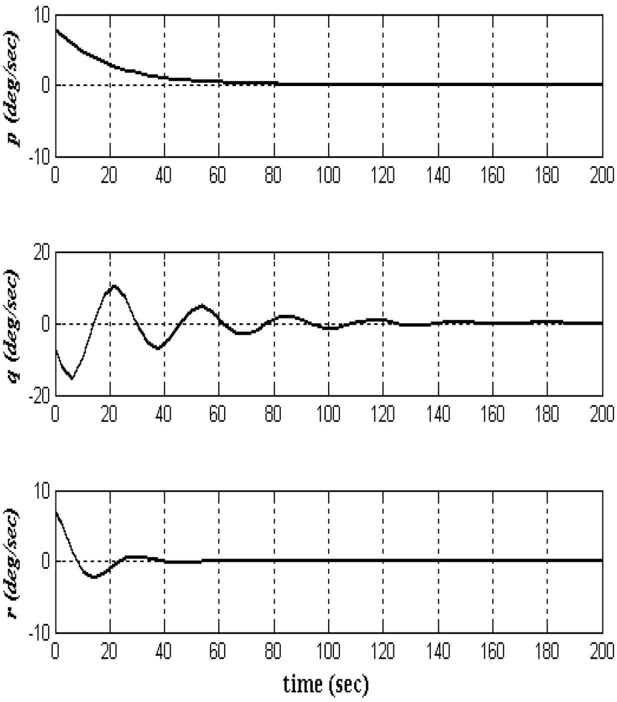

Fig. 1. Time response for p , q and r

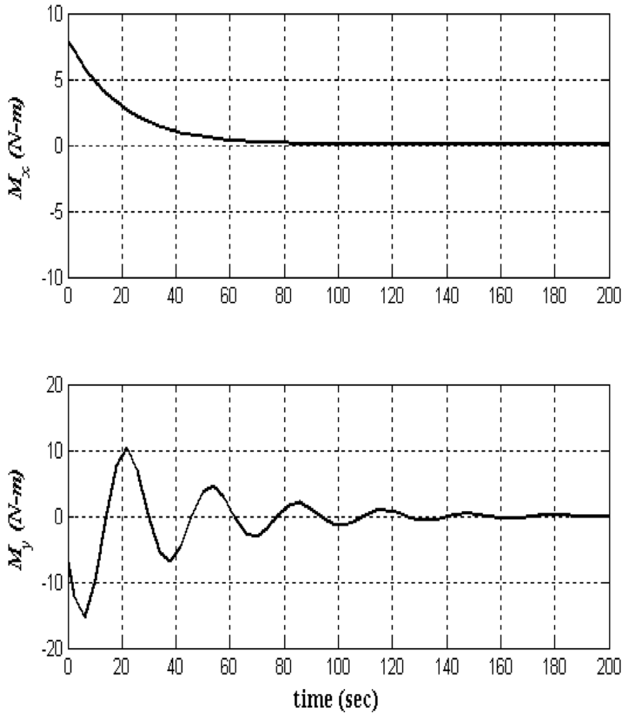

Fig. 2. Time response for M_x and M_y

B. Planar Rigid Body with an Unactuated Internal Degree of Freedom

A planar rigid body with an unactuated internal degree of freedom is a rigid base body that moves on a horizontal plane. An internal degree of freedom is modeled as a single mass particle that is constrained to move along a slot fixed in the base body [12]. A schematic diagram of this system is shown in Fig. 3.

After several transformations [12], the equations of motion can be written in the following form:

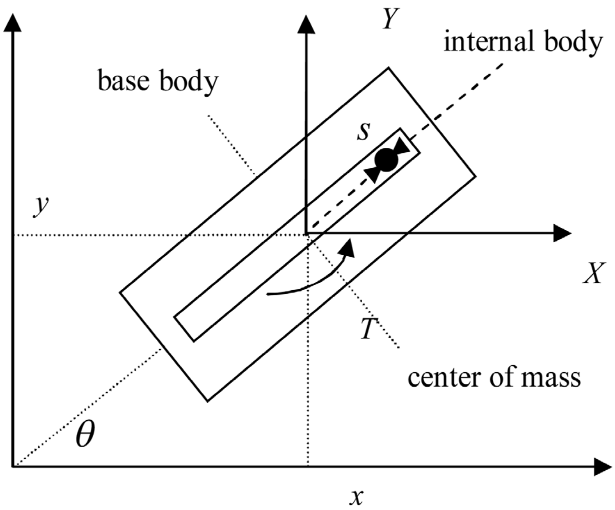

Fig. 3. Planar rigid body with a point mass [18]

$$\begin{aligned}
&\ddot{\theta} = u_1 \\
&\ddot{\xi}_2 = u_2 \\
&\ddot{\xi}_1 = u_3 \\
&\ddot{s} = -u_3 + \xi_2 u_1 + 2\dot{\xi}_2\dot{\theta} + \xi_1\dot{\theta}^2 + s\dot{\theta}^2
\end{aligned} \tag{43}$$

Despite the fact that these equations are not exactly in the form of (3), it is possible to transform them into a form similar to (3) using the following change of variables:

$$\begin{aligned}
&\dot{\theta} = x_1 \\
&\dot{\xi}_2 = x_2 \\
&\dot{\xi}_1 = x_3 \\
&\dot{s} = x_4
\end{aligned} \tag{44}$$

Substituting in (43), the following equations will be generated:

$$\begin{aligned}
&\dot{x}_1 = u_1 \\
&\dot{x}_2 = u_2 \\
&\dot{x}_3 = u_3 \\
&\dot{x}_4 = -u_3 + \xi_2 u_1 + 2x_1 x_2 + \xi_1 x_1^2 + s x_1^2
\end{aligned} \tag{45}$$

The goal of the controller is to stabilize θ, ξ_1, ξ_2, s. Therefore, the following relations are defined:

$$\begin{aligned}
&x_{1,des} = -k_\theta \theta \\
&x_{2,des} = -k_{\xi_2} \xi_2 \\
&x_{3,des} = -k_{\xi_1} \xi_1 \\
&x_{4,des} = -k_s s
\end{aligned} \tag{46}$$

If $[x_1, x_2, x_3, x_4] \to [x_{1,des}, x_{2,des}, x_{3,des}, x_{4,des}]$, according to (44) and (46), θ, ξ_1, ξ_2 and s will be asymptotically stabilized.

If x_2 is taken as the assistant state and assuming:

$$\begin{aligned}
&u_1 = k_1\left(x_{1,des} - x_1\right) \\
&u_3 = k_3\left(x_{3,des} - x_3\right)
\end{aligned} \tag{47}$$

the last equation of (45) will be:

$$\begin{aligned}
&k_4\left(x_{4,des} - x_4\right) = \\
&-u_3 + \xi_2 u_1 + 2x_1\left(ax_{2,1} + bx_{2,2,des}\right) + \xi_1 x_1^2 + s x_1^2
\end{aligned} \tag{48}$$

and therefore, ϕ will have the following form:

$$\phi = \frac{1}{2x_1}\begin{bmatrix} k_4\left(x_{4,des} - x_4\right) + u_3 \ldots \\ \ldots - \xi_2 u_1 - \xi_1 x_1^2 - s x_1^2 \end{bmatrix} \tag{49}$$

According to (26), u_2 will be:

$$\begin{aligned}
&u_2 = k_2\left(x_{2,des} - x_2\right) + \frac{d_1}{c_1 + d_1}\frac{\partial\phi}{\partial x_1}\left(k_1 x_1\right) \\
&+ \frac{d_2}{c_2 + d_2}\frac{\partial\phi}{\partial x_3}\left(k_3 x_3\right)
\end{aligned} \tag{50}$$

Because of being lengthy, the complete formula of u_2 is not presented here.

In order to simulate the closed-loop system response, the following values are considered as the controller coefficients:

$k_1 = 1.83,\ k_2 = 0.9,\ k_3 = 0.52,\ k_4 = 0.88$

$k_\theta = 0.22,\ k_{\xi_1} = 0.14,\ k_{\xi_2} = 0.54,\ k_s = 1.44$

The results are illustrated in Fig.s. 4 and 5.

The values of c_1, c_2, d_1, d_2 are selected as:

$c_1 = 1, d_1 = -0.9, c_2 = 1, d_2 = -0.9$

The controller parameters were obtained by a trial and error procedure. As shown in Figs. 4 and 5, all states have converged into the origin and the controller has demonstrated a good performance. Actually, the presented results are quite comparable with those obtained by [12].

Remark 5: Due to the fact that x_1 appears in the denominator of u_2 in (50), x_1=0 is a singularity point for the controller. In order to avoid this problem, a thin boundary layer (thickness = 0.01) has been defined around x_1. It is assumed that inside this boundary layer, no assistant input is exerted on the system.

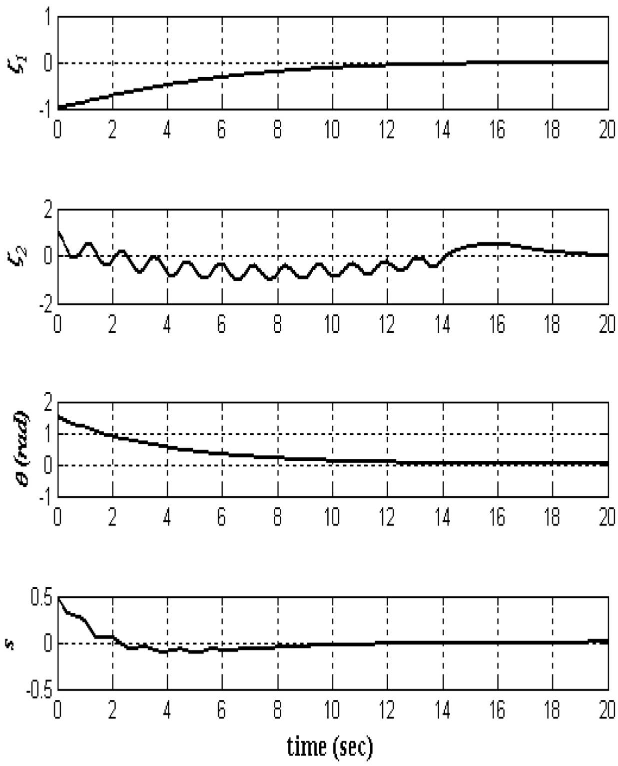

Fig. 4. Time responses for ξ_1, ξ_2, θ and s

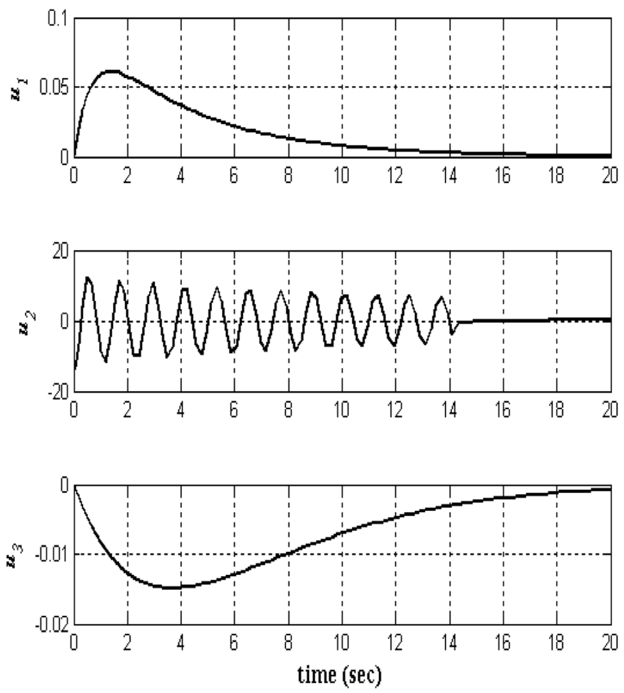

Fig. 5. Time responses for u_1, u_2 and u_3

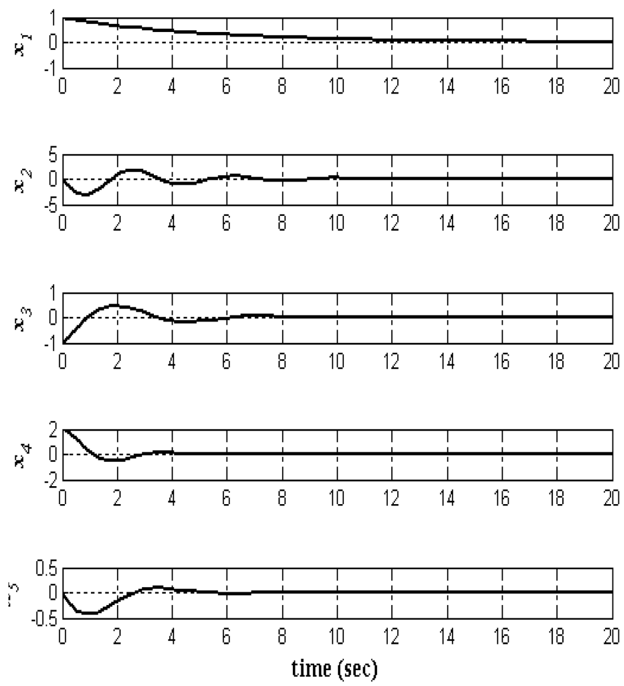

Fig. 6. Time responses for x_1, x_2, x_3, x_4 and x_5

C. Stabilizing an Underactuated System Consisting of Two Underactuated Degrees of Freedom

An underactuated system consisting of the following state-space equations is considered:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_3 \\ \dot{x}_4 &= x_1^2 x_2 \\ \dot{x}_5 &= x_1 x_3 \end{aligned} \tag{51}$$

Pursuing the procedure outlined in section 3, u_1, u_2 and u_3 will be:

$$\begin{aligned} u_1 &= -k_1 x_1 \\ u_2 &= -k_2 x_2 + 2k_1 k_4 \frac{d_1}{c_1 + d_1} \frac{x_4}{x_1^2} \\ u_3 &= -k_3 x_3 + k_1 k_5 \frac{d_2}{c_2 + d_2} \frac{x_5}{x_1} \end{aligned} \tag{52}$$

The controller parameters are selected as:
$k_1 = 0.2, k_2 = k_3 = k_4 = k_5 = 1$
and the results are presented in Figs. 6 and 7, with the values of c_1, c_2, d_1 and d_2 chosen as:
$c_1 = 1, d_1 = -0.9, c_2 = 1, d_2 = -0.9$

Figs. 6 and 7 show a good convergence of responses and a smooth behavior for control inputs. Similar to the first case study, due to the smoothness of the control inputs, no boundary layer is defined in this case.

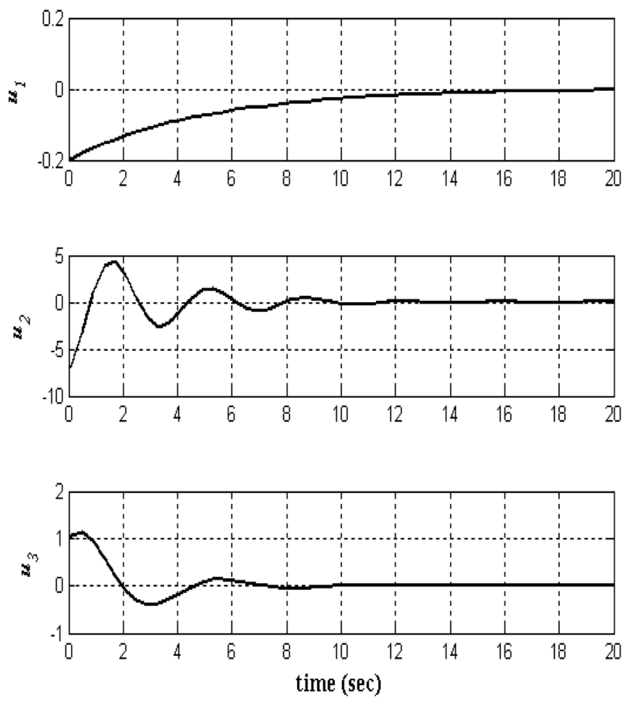

Fig. 7. Time responses for u_1, u_2 and u_3

The latter case study shows the applicability of LSB to underactuated systems with *more than one unactuated state*.

Remark 6: According to [12], the first and second examples are not stabilizable using time-invariant smooth feedback. Therefore, the discontinuous nature of the feedback controls obtained for these systems was quite expectable. Defining boundary layers with appropriate thicknesses will prevent control inputs from being unbounded, especially when the states approach the origin.

4- Conclusions

A new method called "linear state bisection" was introduced to find time-invariant state feedback control laws for a special form of underactuated nonlinear systems. The main idea was to linearly bisect directly actuated states and use them to stabilize the unactuated state/states. The novelty of the proposed method besides its simplicity is that it can be utilized to stabilize underactuated systems with more than one unactuated state. In addition to the advantages proposed by this method, there is still an important challenge that will be considered as a future study: the presence of uncertainties and disturbances.

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APPENDIX A

Insert $\dot{x}_{a_{n2,des}}$ (16) into (14):

$$\dot{z} = cw + \frac{d}{b}\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\dot{x}_{a_i} - a\frac{d}{b}\dot{x}_{a_{n1}} \quad \text{(A-1)}$$

According to (4), $\dot{x}_{a_{n1}}$ will be:

$$\dot{x}_{a_{n1}} = \frac{1}{a}\dot{x}_{a_n} - \frac{b}{a}\dot{x}_{a_{n2}} \quad \text{(A-2)}$$

Now, insert (A-2) in (A-1):

$$\dot{z} = cw + \frac{d}{b}\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\dot{x}_{a_i} - \frac{d}{b}\dot{x}_{a_n} + d\dot{x}_{a_{n2}} \quad \text{(A-3)}$$

According to (6) and (A-3), (17) will be obtained.

APPENDIX B

According to (9):

$$\lim_{t\to\infty} x_{a_{n2,des}} = \frac{-a}{b}\lim_{t\to\infty} x_{a_{n1}} + \frac{1}{b}\lim_{t\to\infty}\phi \quad \text{(B-1)}$$

Since the dynamics of $x_{a_{n1}}$ is:

$$\dot{x}_{a_{n1}} = -k_{a_{n1}}x_{a_{n1}}\ ;\ \lim_{t\to\infty} x_{a_{n1}} = 0 \quad \text{(B-2)}$$

On the other hand, as seen in the simulation section, for the three systems considered, it was possible to adjust the controller parameters such that $\lim_{t\to\infty}\phi = 0$. Therefore, according to (B-1), $\lim_{t\to\infty} x_{a_{n2,des}} = 0$.

APPENDIX C

In order to obtain (21), consider the third raw of (15):

$$\dot{x}_{a_{n1}} = \frac{1}{a}\left(f_{a_n} + g_{a_n}u_n - bw\right) \quad \text{(C-1)}$$

Solve for u_n:

$$u_n = \frac{1}{g_{a_n}}\left(a\dot{x}_{a_{n1}} - f_{a_n} + bw\right) \quad \text{(C-2)}$$

Now, insert w from (20), in this equation:

$$u_n = \frac{1}{g_{a_n}}\begin{pmatrix} a\dot{x}_{a_{n1}} - f_{a_n} - \frac{bk_z}{c+d}z\ \ldots \\ \ldots - \frac{d}{c+d}\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\dot{x}_{a_i} + \frac{d}{c+d}\dot{x}_{a_n}\end{pmatrix} \quad \text{(C-3)}$$

Consider *lemma 1*, (13) and (22):

$$u_n = \frac{1}{g_{a_n}}\begin{pmatrix} -ak_{a_{n1}}x_{a_{n1}} - f_{a_n} - \frac{bk_z c}{c+d}x_{a_{n2}}\ \ldots \\ \ldots - \frac{d}{c+d}\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_{a_i}}\dot{x}_{a_i} - \frac{dk_{a_n}}{c+d}x_{a_n}\end{pmatrix} \quad \text{(C-4)}$$

Considering assumption 2 and (4), (C-4) will be simplified to (21).

APPENDIX D

Substitute (20) and (23) in (18):

$$\begin{aligned} \dot{x}_{a_1} &= -k_1 x_{a_1} \\ \dot{x}_{a_2} &= -k_2 x_{a_2} \\ &\ \vdots \\ \dot{x}_{a_{n1}} &= \frac{1}{a}\left(-k_{a_n}x_{a_n} + \frac{bk_z}{c+d}z - \frac{d}{c+d}\dot{x}_{a_n}\right) \\ \dot{z} &= -k_z z \end{aligned} \quad \text{(D-1)}$$

Considering (13), (22) and *lemma 1*, $\dot{x}_{a_{n1}}$ will be:

$$\dot{x}_{a_{n1}} = \frac{1}{a}\left(-k_{a_n}x_{a_n} + \frac{bk_z c}{c+d}x_{a_{n2}} + \frac{d}{c+d}k_{a_n}x_{a_n}\right) \quad \text{(D-2)}$$

After performing some mathematical operations and eliminating $x_{a_{n2}}$, $\dot{x}_{a_{n1}}$ will be simplified to the following form:

$$\dot{x}_{a_{n1}} = -k_{a_{n1}}x_{a_{n1}} \quad \text{(D-3)}$$