AUT Journal of Modeling and Simulation

AUT J. Model. Simul., 58(1) (2026) 33-66
DOI: 10.22060/miscj.2026.25114.5455

Proposing a Gaussian-Curvature-Based Nonlinear Natural Gas Pricing Model Incorporating Weather, Consumer Income, and Demand Dynamics

Seyed Hossain Ebrahimi *, Abdolhamid Safaei Ghadikolaei, Seyed Hamid Hashemi Petroudi

Faculty of Economics and Administrative Sciences, University of Mazandaran, Babolsar, Iran.

ABSTRACT: Nowadays, natural gas pricing, particularly for the residential sector, has always been a challenging issue in the natural gas market due to the rapidly changing factors affecting natural gas supply and demand mechanisms. Weather temperature and demand are the most important and influential factors that have repeatedly attracted scholars to mathematically model natural gas prices, especially for forecasting purposes. Traditionally, the prices of other major substitute energy sources, such as oil, gas oil, and coal, have also been formally used in natural gas pricing mechanisms across different natural gas markets worldwide. However, there are almost no studies that specifically discuss natural gas pricing based solely on weather temperature, demand, and residential end users' average income simultaneously. In this study, we are determined to model natural gas prices in terms of weather temperature and demand for different income levels using a curvature-based formula under two scenarios. The main strategies underlying this research are that higher prices should be paid when the weather temperature decreases and the demand increases, respectively. In addition, we deliberately reformulate the proposed basic model (Scenario One) using the income elasticity of demand to define natural gas prices for different demand levels. There are two notable features of this study: (1) the proposed mathematical pricing models are sufficiently flexible to simultaneously incorporate weather temperature, demand, and average income to categorize residential end users; and (2) the curvature formula provides the flexibility to reconfigure the proposed models into alternative natural gas pricing equations, which can strategically benefit policymakers in the natural gas market.

Review History:

Received: Nov. 15, 2025
Revised: Jul. 01, 2026
Accepted: Jul. 14, 2026
Available Online: Jul. 20, 2026

Keywords:

Natural Gas Price
Weather Temperature
Income
Mathematical Modeling
Curvature Formula

1- Introduction

Natural gas ranks among the fastest-growing energy sources in the world and is regarded by many in the energy industry as a comparatively clean, low-cost, and versatile fuel. Nowadays, natural gas, which is recognized globally as a clean source of energy [1] and a strategic fuel [2], is commonly used in the residential and industrial sectors. The widespread use of natural gas, together with its environmentally friendly characteristics and economic advantages [3], has encouraged gas companies on the supply side of the natural gas production system to carefully consider their selling prices [4]. Moreover, natural gas pricing [5,6] is one of the most important issues in today's global energy markets due to the key role of natural gas as a major and efficient energy resource, particularly for heating purposes in power plants [7] and household heating appliances [8]. With this in mind, natural gas prices are not stable over time and vary considerably across global markets. It is also worth mentioning that natural gas pricing frameworks are often influenced by the prices of other major fossil fuels, such as crude oil and coal. Additionally, the volatility of international energy markets, which is primarily affected by natural gas supply costs, environmental restrictions, climate change [9], international policies, economic uncertainties, and demand-side factors, must be carefully considered when evaluating natural gas prices at both the local and global levels. Research on natural gas pricing mechanisms has been widely conducted by both practitioners and academic scholars. From a technical and mathematical perspective, there are two main pricing mechanisms in natural gas markets [10]. Oil indexation and hub pricing [11,12] are the two principal natural gas pricing mechanisms used in gas markets. Under the oil indexation mechanism, crude oil prices are generally assumed to be the primary basis for natural gas pricing. In contrast, under the hub pricing mechanism, natural gas is typically priced competitively based on supply and demand determinants. There are three major natural gas markets, namely the European Union, Japan, and the United States [13]. However, these major gas markets are, in reality, closely integrated with the global oil market [14–16]. Furthermore, seasonality and weather temperature [17–22], financial speculation [23], and financial markets [24–27] have significant and unavoidable

*Corresponding author's email: h.ebrahimi@stu.umz.ac.ir

impacts on natural gas price fluctuations. On the other hand, the increasing rate of natural gas consumption has made it more difficult to accurately analyze supply capacity, production rates, demand levels, and price irregularities. Therefore, natural gas, as the cleanest low-carbon primary energy source [28], which has been rapidly adopted in the residential sector and domestic power plants worldwide, should be priced in an effective manner that covers all costs and risks associated with natural gas supply while simultaneously maintaining the intended level of demand. However, natural gas prices have recently been revised and restructured for the residential sector in several regions around the world. Furthermore, because of its strategic importance, environmentally friendly low-carbon characteristics, and widespread use in urban areas, particularly for heating purposes, natural gas has recently been priced higher for residential consumers than for commercial and industrial users in some countries, such as the United States (EIA, Natural Gas Monthly), whereas the opposite pricing structure exists in other countries, such as Germany (https://www.statista.com). In particular, weather temperature is generally regarded as an indispensable parameter and one of the most important factors affecting energy markets, especially natural gas markets. This phenomenon is commonly referred to as the seasonal impact on energy markets. Furthermore, natural gas price volatility caused by daily temperature variations has been extensively investigated and modeled by many researchers [29–33]. These studies clearly demonstrate that the behavior of natural gas prices with respect to weather temperature can be more effectively analyzed and understood. Natural gas prices and consumers' income are commonly formulated with respect to natural gas demand to mathematically explain changes in consumption levels. Price elasticity and income elasticity of demand are two econometric measures used to quantify the responsiveness of the quantity of natural gas demanded to changes in price and income, respectively. Additionally, these elasticities are very useful tools for policymakers in energy markets [34–36]. Moreover, several previous studies conducted in different countries have demonstrated that price and income elasticities can be used to analyze natural gas demand over specific time periods [37–43].

Basically, price and income, which are considered part of the socioeconomic determinants, together with temperature, which is recognized as one of the major meteorological variables [44], are among the most important and logical measures for studying natural gas markets in greater detail. Therefore, in the present paper, we deliberately aim to reexamine and model natural gas prices, particularly for the local residential sector, by considering only weather temperature and the end users' average income.

The remainder of this paper is organized as follows. Section 2 reviews previous studies investigating the relationships among weather temperature, end users' average income, and natural gas prices. In addition, the mathematical models describing these relationships are presented. The research gap is discussed in Section 3. Section 4 presents the research methodology and the mathematical strategy used to formulate the natural gas pricing model. Section 5 develops the proposed mathematical model. Finally, Sections 6 and 7 present a numerical case study and the conclusions, respectively.

2- Literature review:

2- 1- Review scheme:

At this point, based on the previous studies, we are determined to separately identify clear evidence to further corroborate the effects of weather temperature and income on the natural gas pricing mechanism. Eventually, in the last two parts of this section, we review the studies related to mathematical models of natural gas pricing. Accordingly, the systematic review strategy of PRISMA (Preferred Reporting Items for Systematic Reviews and Meta-Analyses) has been adopted to select and screen the relevant articles. The two well-known scholarly databases, ScienceDirect and Google Scholar, were searched for studies published before September 2024 based on the eligibility criteria, including weather temperature, natural gas price, income, natural gas demand, and mathematical models of natural gas pricing. Ultimately, the initial 361 selected research articles were systematically screened to identify the 14 articles included in this study. The research screening process is illustrated in Figure 1.

2- 2- Origins and nature of weather temperature affecting natural gas price:

Among all factors, weather information is widely recognized as having an important effect on the natural gas market [45]. Exploring the relationship between natural gas prices, the energy stock market, and daily temperature has attracted researchers and academics to comprehensively model natural gas price volatility and its fluctuations. Hence, there are many studies that exclusively discuss the significant effect of weather temperature on natural gas prices [19–21,29,33,46–54]. The major findings of these studies show that natural gas prices and their volatility can be more explicitly explained by weather temperature. Generally speaking, there is a close relationship between weather temperature and natural gas prices, which leads to volatility in the natural gas market [45]. Weather temperature is also reported by the U.S. Energy Information Administration [55] as a major demand-side factor affecting natural gas prices. Cold and hot weather significantly influence natural gas demand and supply, which inherently cause changes in natural gas prices. Therefore, due to weather variations, natural gas prices are highly volatile [56].

2- 3- Role of end users' income affecting natural gas price:

Urban natural gas demand is usually largely affected by income in regions with low natural gas prices [44]. As far as we know, end users' income is still being used by practitioners to formulate the well-known measure of income elasticity. Income elasticity of demand, which describes the responsiveness of the quantity of natural gas demanded to changes in income, is often integrated with natural gas prices to thoroughly investigate natural gas consumption

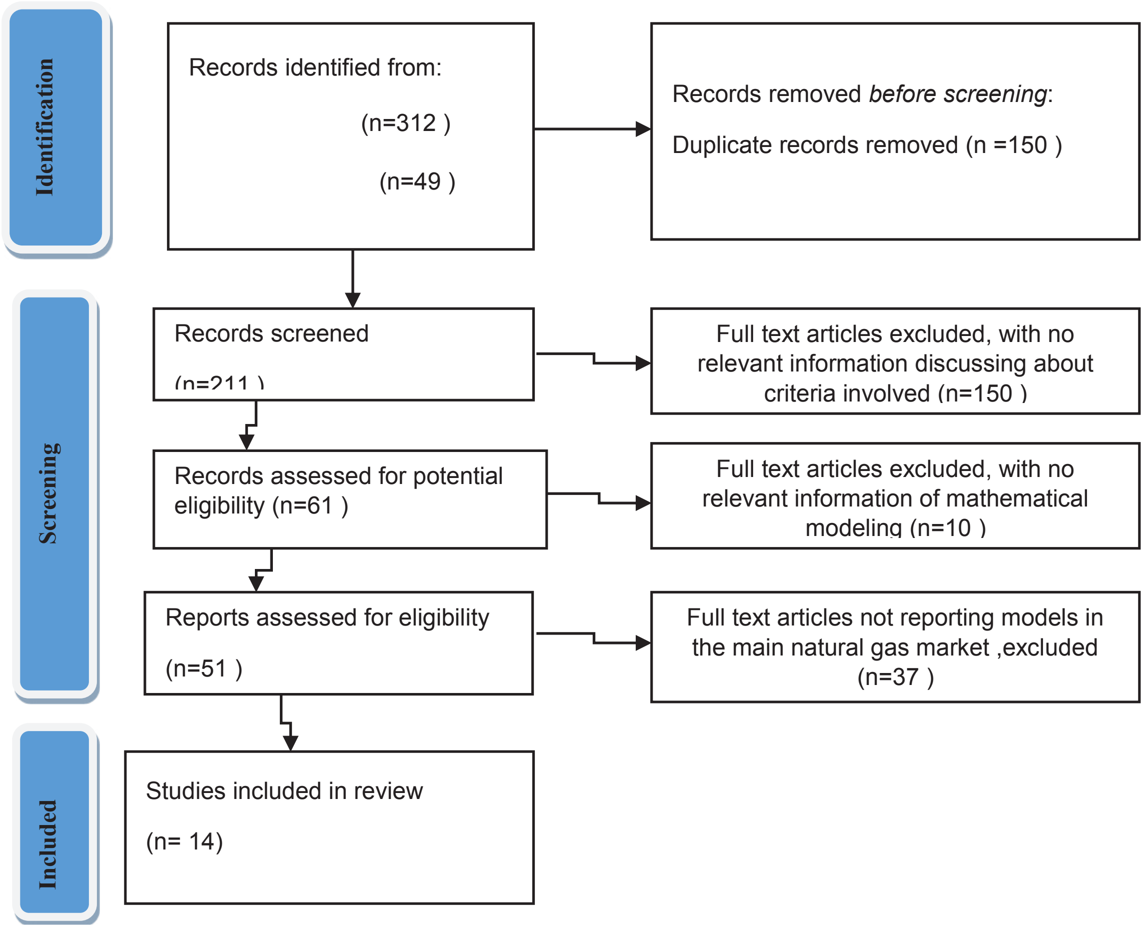

Fig. 1. Flow chart diagram of article screening.

trends [44,57–60]. Consumers with higher incomes are more inclined to consume more natural gas [44] than those with lower incomes. Thus, pricing policies and other regulatory strategies related to natural gas prices can be highly effective in controlling natural gas consumption. Consumers in the urban sector are sensitive to price; therefore, an increase in the urban natural gas price will reduce final consumption [44]. Moreover, residents' price affordability [61] and income have been investigated in several studies [61–64]. The main finding of these studies is that residents with higher incomes are willing to pay higher prices [65]. Specifically, the tiered gas pricing (TGP) policy has been implemented nationwide in the residential sector of China since January 2016. In other words, tiered pricing may impose only limited constraints on wealthy households. This concern may affect the effectiveness of the tiered pricing policy. This phenomenon can be defined as the income effect on tiered pricing [65]. In general, the effect of income, as a major influencing factor, cannot be ignored when investigating natural gas demand and establishing natural gas prices.

2- 4- Generally applicable natural gas pricing mathematical models:

Prices play a crucial role in natural gas trade [56]. Natural gas, as one of the most important commodities in the energy market, has been priced using two chief different approaches. Direct regional market pricing (which will be discussed in more detail in Section 2.4) is based on some distinguished factors like oil price [16] and LNG, which is known as a classical mechanism throughout the world, and time series forecasting structures that usually include specified parameters and determinants like temperature, regional supply, demand, and storage capacity [24,66-68]. There are many extant and valuable mathematical models adopting various techniques to systematically and precisely address natural gas prices. The vector error correction model (VECM[1]) is a mathematical model that has been used widely

1. A Vector Error Correction Model (VECM) identifies and characterizes the co-integrating relationship(s) between two or more sets of time series data. The relationship is measured in the level of each time series variable. For both crude oil and natural gas, it also models the *change* in price for each as

to analyze the long-run relationship between natural gas prices and crude oil prices. VECM methodology has been deployed to study the Henry Hub natural gas prices and global crude oil prices in the US by Brown and Yücel [67] and Ramberg and Parsons [68]. One of the most beneficial features of VECM methodology is the ability to simultaneously consider the short-term influence of various factors, including the volume of natural gas production and total gas imports, weather variables (cooling degree days [CDD] and heating degree days [HDD]), industrial growth rate, and the Henry Hub natural gas prices [69]. Natural gas pricing in Europe (e.g., Turkey) is usually calculated by a direct regression model by applying some main fuel alternatives and competing energy prices such as Mazut or gas oil [70]. Yorucu and Bahramian [71] modeled the natural gas price based on crude oil price and taxation data for EU-12 countries including Austria, France, Greece, the Netherlands, Spain, the UK, Denmark, Czech Republic, Finland, Poland, Slovakia and Ireland. They studied the cointegration relationship between natural gas price, crude oil price, and taxation using Pedroni's cointegration model and a simple regression model. Acaravci et al. [72] also analyzed the long-run equilibrium relationship between natural gas prices and stock prices among the EU-15 countries. They employed vector error correction (VEC) modeling, Granger causality test,[1] and cointegration test to estimate the interconnection perceived. An autoregressive distributed lag modelling (ARDL) methodology has been used to model imported natural gas prices to Turkey by Yorucu [74]. This investigation showed that taxation and exchange rates would be the two significant determinants affecting the natural gas price imported to Turkey. In a recent comprehensive study, Ribeiro et al. [75] studied about short run relation between natural gas price and liquefied natural gas (LNG) imports from Nigeria. They examined short-term and long-term causal dependencies between natural gas price and LNG imports from Nigeria, deploying a VAR-VECM panel data model analysis for 34 spatial units from Europe over 31 units of time representative of the period 2007–2022. Ergen and Rizvanoghlu [76] investigated the volatility of natural gas prices associated with natural gas futures contracts traded on the New York Mercantile Exchange (NYMEX), using augmented GARCH[2] methodology to check the impact of storage level, time to maturity, and weather temperature. They made use of time series price data of 155 contracts. In another study, Ling, et al. [45] studied natural gas price volatility against extreme weather indicators, including wind speed, temperature, and precipitation information by extended GARCH-MIDAS-ES[3] model. They concluded that the predictive model adding weather indicators can indeed outperform the model without weather indicators [45]. Xu and Lien [80] compared and evaluated different methodologies of natural gas and crude oil price forecasting volatilities established by GAS (Generalized Autoregressive Score), GARCH, and EGARCH models. This study aims to technically address the performance of GAS, GARCH, and EGARCH mathematical models. Additionally, the vector autoregressive model (VAR) has been applied to comprehensively study the natural gas market in Germany by Nick and Thoenes [81]. Their results showed that the natural gas price is structurally affected by temperature, storage, and supply shortfalls in the short term, while the long-term development is closely tied to both crude oil and coal prices [81]. Gao et al. [66] used three categories of flexible time-varying parameter models to better understand and assess the impacts of regional features and structural instability on natural gas price forecasts in three important natural gas markets of the US, EU, and Japan. Their work profoundly determined the role of time variation of data in variances for forecasting performance. In total, they observed that models with Markov switching variance often improve the forecast accuracy for the US market. For the EU and Japanese markets, models with stochastic volatility are generally the best models to forecast natural gas prices [66]. In another study, Su et al. [82] decided to adopt the least squares regression boosting (LSBoost[4]) algorithm as a new machine learning approach to forecast the natural gas spot prices of the Henry Hub market. Their results revealed that the LSBoost algorithm has better performance compared to linear regression, linear support vector machine (SVM[5]), quadratic SVM, and cubic SVM. Alam et al., [86] investigated thoroughly the price of natural gas along with the price of oil and coal in India for two scenarios of pre-and post-COVID-19 pandemic using Autoregressive Integrated Moving Average (ARIMA[6]), Simple Exponential Smoothing[7], and K-Nearest Neighbor[8] approaches. The outcome of their work clearly showed that the Autoregressive Integrated Moving Average technique would be suitable for predicting India's oil, coal, and natural gas prices. Having ended this paragraph and for simplicity and a brief description at a glance, the mentioned existing research on natural gas pricing mathematical models are summarized in Table (1).

a function of *past* changes in the prices of both commodities. This is done to capture the effects of long-lived shocks to the price series [68].

1. The Granger causality test is a statistical hypothesis test for determining whether one time series is useful in forecasting another time series [73] .

2. Generalized Auto Regressive Conditional Heteroskedasticity (GARCH), the statistical model helps analyze time-series data where the variance error is believed to be serially autocorrelated [77].

3. For more information about GARCH-MIDAS-ES methodology, refer to [78,79].

4. For more information about least squares regression boosting, refer to [83,84].

5. For more information about support vector machine, refer to [85].

6. ARIMA approach as one of the most popular forecasting models presented by Box and Jenkins being used for mathematical analysis that involve time-sequence datasets. For more information, refer to [86].

7. SES is a prediction technique that uses past time-series data with a horizontal approach rather than a seasonality or trend. SES uses a single smoothing coefficient (alpha). For more information, refer to [86].

8. For more information about K- Nearest Neighbor approach, refer to [86].

Table 1. The main existing literature discussing natural gas pricing mathematical models. (Continued)

Authors/ Reference	Country/ Regions	Research scope	Type of mathematical model	Period	Factors	Main findings/Highlighted aspects
Brown and Yücel, 2008 [67]	U.S	Industrial	Simple regression analysis/ Error-correction model	1994-2006	Crude oil price , weather, Seasonality, natural gas storage conditions, Disruptions in natural gas production	A continuum of prices at which natural gas and petroleum products are substitutes
Ramberg and Parsons, 2012 [68]	U.S	Industrial	Vector error correction model	1991-2010	Crude oil price, Typical seasonality, Unseasonably cold or warm weather, Shut-in production due to hurricanes, storage levels	Despite large temporary deviations, Natural gas prices continue to exhibit evidence of a cointegrating relationship with crude oil prices
[70]	Turkey	Gas exportation	Regression	-	Mazot price or Gas oil price	Natural gas price is usually indexed on petroleum products
Yorucu and Bahramian, 2015 [71]	EU-12 countries	Internationally traded gas price	Pedroni's cointegration model and Regression	2001-2012	Crude oil price, Taxation data	A significant long-run relationship between the prices of natural gas, crude oil and taxation
Acaravci et al., 2012 [72]	EU-15 countries	Regional natural gas prices	Vector error correction model, Granger causality test and Cointegration test	1990-2008	Stock prices	Positive correlation between an increase in natural gas prices and industrial production growth, which in turn affects stock returns.
Yorucu, 2014 [74]	Turkey	Gas importation	Autoregressive distributed lag modelling (ARDL)	1988Q1–2012Q4	Taxation, Exchange rates	The crude oil price does not have any important influence on natural gas prices, while taxation and exchange rates have a significant effect on imported natural gas prices
Ribeiro et al.,2023 [75]	EU-34 countries	LNG importation	Vector autoregressive (VAR), Vector error correction model (VECM)	2007–2022	LNG imports from Nigeria	34 European countries' natural gas market have been analytically categorized in short and long term relationship due to LNG imports from Nigeria.
Ergen and Rizvanoghlu, 2016 [76]	U.S	gas futures contracts traded on the New York Mercantile Exchange (NYMEX)	GARCH methodology	January 2001 - November 2013	Storage level, Time to maturity, and Weather temperature	Using the augmented GARCH models, the storage-based model performs better than the weather-based model

Table 1. The main existing literature discussing natural gas pricing mathematical models.

Ling, et al., 2022 [45]	U.S	natural gas market in New York City	Extended GARCH-MIDAS-ES model	January 1995 to July 2021	Weather indicators including wind speed, temperature, and Precipitation information	Extreme weather information can better explain natural gas price volatility
Xu and Lien, 2022 [80]	China market, U.S market, Russia market, and Canada market	natural gas market	GAS (Generalized Autoregressive Score), GARCH, and EGARCH models	November 23, 2009, to August 11, 2020	Crude oil spot price, natural gas spot price	Distinct forecasting performance between GAS and GARCH-type models for crude oil and natural gas across different horizons
Nick and Thoenes, 2014 [81]	Germany	German natural gas market	Vector autoregressive (VAR)	January 2008 to June 2012	Temperature, storage and supply shortfalls	Natural gas price is affected by temperature, storage and supply shortfalls in the short term
Gao, et al., 2021 [66]	U.S, EU and Japan	Regional natural gas market	Autoregressive (AR) model, Time-varying parameters with stochastic volatility (TVCSV) models, Markov switching models (MS)	1992 - 2019	US gas price, European gas price, Japan gas price	Models with time-varying covariance are crucial in predicting the gas prices across the three markets, Time-varying dynamics governing the AR coefficient, e.g., TVC, does not seem to play an important role in improving forecasting accuracy for the price of natural gas, Models that include heavy-tailed distributions, e.g. t innovations, exhibit substantial improvements over the benchmark AR with a constant volatility and normal innovations
Su et al., 2019 [82]	U.S	Henry Hub natural gas market	Least squares regression boosting (LSBoost) algorithm	January 2001 to December 2017	Historical price data of Henry Hub	The LSBoost algorithm has better performance compared to linear regression, linear support vector machine (SVM), quadratic SVM, and cubic SVM
Alam et al., 2023 [86]	India	Regional natural gas market	Autoregressive Integrated Moving Average (ARIMA), Simple Exponential Smoothing, and K-Nearest Neighbor	January 2020 to May 2022	Daily oil, coal, and natural gas closing prices of the NIFTY 500 Index	Autoregressive Integrated Moving Average technique would be suitable for predicting India's oil, coal, and natural gas prices

Integrated and combined usage of methods of natural gas pricing has increased significantly in the past decades. Recently, a growing body of research involving natural gas pricing models has been dedicated to hybrid models comprising classical mathematical methods (like ARIMA and GARCH) and new complex machine-based methodologies, including artificial neural networks, deep learning algorithms, and Convolutional neural networks. Machine learning algorithms have gradually become popular tools for natural gas price forecasting [87]. In this regard, Saghi and Rezaee [88] and Nguyen and Nabney [89] used various combining forms of artificial neural networks (ANNs), support vector regression, and generalized autoregressive conditional heteroscedasticity (GARCH) for natural gas price forecasting. Pei et al., [90] presented a novel model employing the temporal convolutional network (TCN[1]) and dynamic learning rate[2] for natural gas price forecasting for the next two weekdays. Herrera et al., [92] provided a profound comparison between traditional econometric methodologies and machine-based learning techniques, which were applied for natural gas and oil price prediction as well. Moreover, there are many other studies [93-102] that have made use of complicated hybrid models and various deep learning methods[3] and different kinds of artificial neural networks for making the natural gas forecasting system more accurate. The fundamental goal of these techniques is to extract knowledge from the noisy and chaotic nature of time-series data [104], especially those relevant to natural gas prices.

2- 5- Regional applicable natural gas pricing mathematical models

As aforementioned stated, in some regions of the world, natural gas (particularly its liquefied form, namely LNG) sold to domestic consumers or exported has been specifically and directly priced based on some important substitutional fuels' prices. This tricky issue is vital and inevitable, particularly for countries that import fuels like LNG or crude oil. In many parts of the world, the import price of natural gas has historically been linked to the price of refined oil products [105]. On the other hand, different natural gas markets around the globe have their own geo-econometric rules due to financial complexities and governmental interference [65] associated with. Thus, prices and pricing mechanisms vary across regions [106]. Contrary to north America natural gas market, which almost only deploys a gas-on-gas competition strategy for natural gas pricing mechanisms, some European gas prices and also the relatively more Asian natural gas market are fundamentally linked to crude oil prices, known as oil indexation strategy (Figure 2). Complementarily, [106] reported that "U.S gas prices are set through gas-on-gas competition in the Henry Hub, whereas European gas prices are mixed-hub prices from the UK's National Balancing Point (NBP[4]) and the Dutch Title Transfer Facility (TTF[5]) as well as indexation to oil prices. Gas prices in Asia are usually indexed to the Japanese Customs-Cleared (JCC[6]) oil price, the Brent oil price, or the official Indonesian Crude Price. So, at this point, we are to revisit some simple applicable regional formulas engaging the historical natural gas- crude oil relationship.

The classical relationship between liquefied natural gas and crude oil in the Japanese market can be mathematically described as shown in linear formula (1).

$$P_{LNG} = \alpha + \beta \bar{P}_{JCC} + \varepsilon \qquad (1)$$

where P_{LNG} stands for the import price for LNG in Japan and $\bar{P}_{JCC}$ would be the 6-month average price of JCC. The constant α captivates the degree of risk aversion negotiated and discussed between importers and exporters, meanwhile the coefficient β determines sensitivity to oil prices [105]. It would be notable to express that the coefficients of formula (1) can be estimated using the ordinary least squares regression method by applying past data.

The natural gas price in Russia has also strongly dependent on the particular conditions relevant to each contractual terms negotiated by importers and exporters for a valid duration of specified period of time. Generally, Russian natural gas pricing mathematical formula are traditionally and structurally based on fuel oil and the gasoil futures free on board (FOB) Mediterranean prices [108]. The more usually accepted representation of Russian natural gas quarterly clearing price is given by formula (2).

$$P_{RUS_t} = \alpha + \beta P_{RUS_{t-1}} \, {}^{1}\!/_{2} \left(\frac{\bar{F}_t}{\bar{F}_{t-1}} + \frac{\overline{G_t}}{\bar{G}_{t-1}} \right) + \varepsilon_t \qquad (2)$$

1. TCN is a relatively recent time series forecasting model that performs exceptionally well while processing time series [91].
2. The dynamic learning rate setting further improves the model's predictive ability. The dynamic learning rate overcomes the problem of the model failing to converge due to a high learning rate, and the model easily falls into a locally optimal solution owing to a low learning rate, allowing the model to identify the optimal solution faster and with greater stability [90].
3. Deep-learning methods are representation-learning methods with multiple levels of representation, obtained by composing simple but non-linear modules that each transform the representation at one level (starting with the raw input) into a representation at a higher, slightly more abstract level. With the composition of enough such transformations, very complex functions can be learned [103].

NBP is the leading traded benchmark for natu-. 4
ral gas in Europe; even spot LNG trading is often
.[linked to the NBP price [108

5 .The Title Transfer Facility, which more commonly known as TTF, is a virtual trading point for natural gas in the Netherlands and has been set up by Gasunie in 2003. Gasunie is a natural gas company located in Dutch. For more information refer to the site: https://www.gasunietransportservices.nl/en.

6 .The JCC is the average price of customs-cleared crude oil imports into Japan and is published by the Petroleum Association of Japan [109].

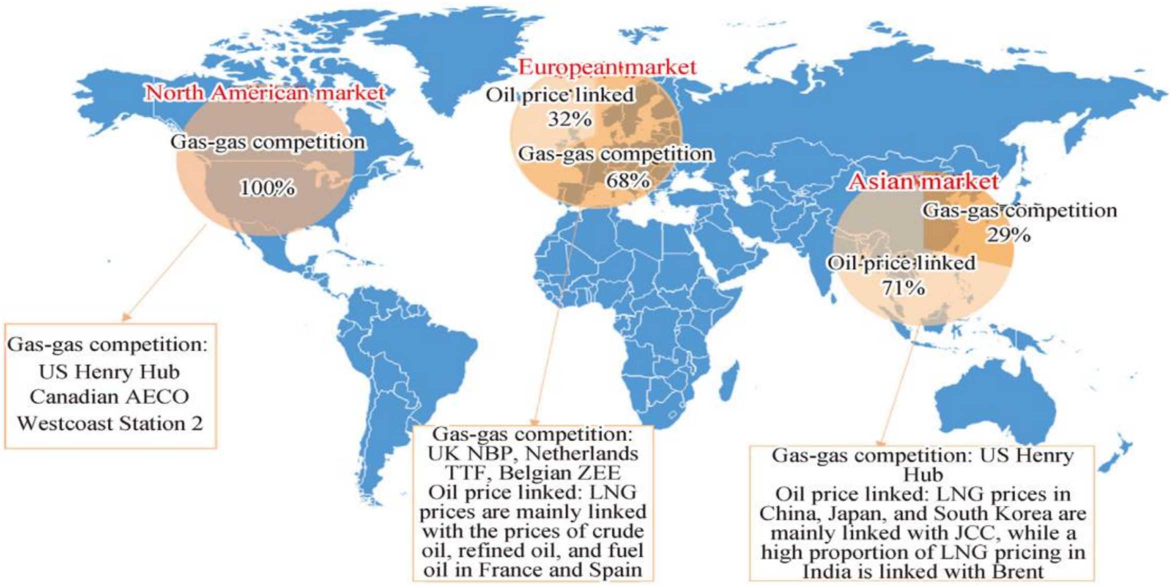

Fig. 2. LNG pricing mechanisms were formed in three major markets.(Source: [107]).

where P_{RUS_t} is the quarterly price of Russian natural gas, α is a constant, F_t is the price for fuel oil over the last 6 months, G_t is the average price of gas oil over the last 6 months, and ε_t is the error term [105]. The coefficients in formula (2) are usually estimated by regression techniques.

In North America, before the natural gas market liberalization and at least short-term decoupling of natural gas and oil prices [33,68,110,111] in the early 2000s, which was fundamentally caused by seasonal factors—such as inventory, weather, and supply shocks [106], the Henry Hub natural gas price (P_{HH}) and crude oil price (P_{Oil}) have been linearly and traditionally formulated as shown in equation (3).

$$P_{HH} = \alpha + \beta P_{Oil} + \varepsilon \qquad (3)$$

As can be seen, there has been no single unified formula established and used everywhere and every time in the three major natural gas markets in the world. It mainly returns to the inherent characteristics of natural gas markets, including regional supply and demand levels, market structure, long- and short-term pricing contract conditions, governmental policies, and, importantly, other energy prices like oil and gasoil.

3- Research gap

Taking a brief but concise look at the previous studies, it can be observed that there are very few investigations examining the direct and comprehensive influence of income on natural gas pricing mechanisms. Besides, the effect of weather temperature has mostly been applied to specifically and directly explain natural gas demand behavior, which is commonly known as the seasonality effect in the natural gas market. Additionally, weather temperature, alongside other factors such as crude oil prices and storage levels, has usually been employed to forecast natural gas prices on a daily or monthly basis. Consequently, in the present study, we strategically reconstruct the natural gas pricing mechanism for the residential sector by exclusively considering the effects of weather temperature, demand, and income variables in such a way as to indirectly manage natural gas consumption to some extent. On the other hand, in our research, it has been decided to dynamically define natural gas prices as a function of weather temperature and natural gas demand for different levels of end users' average income.

4- The proposed strategies and research methodologies

In this study, we are going to establish different forms of natural gas pricing mechanisms using curvature-based mathematical formulas. Applying curvature equations will automatically give us the flexibility to obtain versatile natural gas pricing formulas through different reconfigurations of the required parameters. More importantly, our proposed strategies and initial assumptions, which are intended to define natural gas prices, can also be easily incorporated by applying curvature formulas. It should be noted that the proposed strategies are represented in this investigation by differential mathematical forms. Furthermore, it should also be mentioned that the two-dimensional curvature formula and the Gaussian curvature equation are specifically used in this research for two distinct scenarios. Simply stated, Scenario One describes natural gas prices in terms of weather temperature, whereas Scenario Two describes natural gas prices in terms of weather temperature and demand. The strategies and methodologies used in the present research are described in detail in the following two scenarios.

4- 1- Scenario one

4- 1- 1- Strategies implied

Based on our review, the most influential macroeconomic factors and the most effective uncertainties driving the natural gas price are depicted in Figure (3), and for the present study, we focus on two fundamental factors, including weather temperature and income from the demand side, affecting the natural gas price. Furthermore, as previously stated in the literature part, weather temperature usually causes in increasing of natural gas consumption (*COVARIANCE(weather temperature, natural gas consumption)<0*), which finally results in higher prices set by natural gas companies (*COVARIANCE (natural gas consumption, natural gas price)>0*), especially for the residential sector. So in this work, we are determined to strategically relate the higher price being paid by residential end users to weather temperature as being got colder. This strategy is intentionally taken into account to financially manage the consumption of natural gas. Thus, first of all, we mathematically consider the non-linear inverse relationship between natural gas price and weather temperature to apply the proposed strategy. Secondly, it should be noted that there are other important surreptitious assumptions used to build the research's mathematical model and also for approximately rationalizing the natural gas consumption. The assumptions are as follows:

Assumption 1:

"*.Residential end users who have higher income tend to consume more natural gas*. (*COVARIANCE (income, natural gas consumption)>0*)"

Assumption 2:

"*.Residential end users who have higher income affordability, tends to pay higher price*. (*COVARIANCE (income, price)>0*)"

For simplicity and being better perceived, the casual loop diagram of our study's factors are drawn in figure (4).

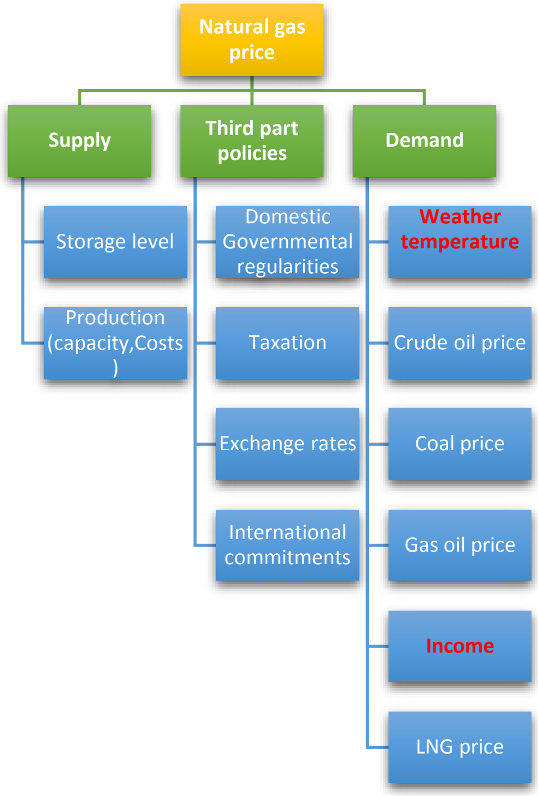

Fig. 3. Schematic layout of the most influential and predominant factors affecting the natural gas price mechanism.

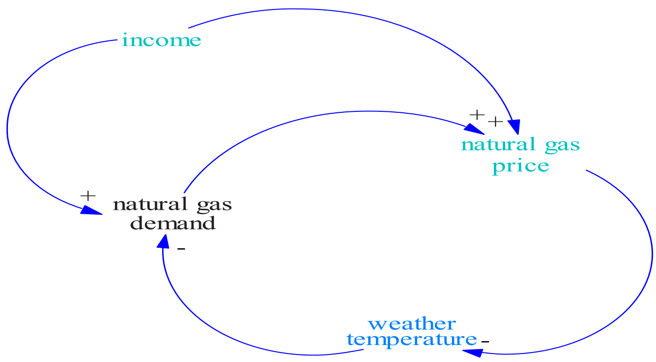

Fig. 4. The causal loop diagram of our study's factors.

4- 1- 2- Methodology

Differential equations are one of the most commonly used mathematical tool which has been adopted to analyze the different features of a system being modeled. When mathematical modeling is used to describe physical, biological, or chemical phenomena, one of the most common results is either a differential equation or a system of differential equations, together with appropriate boundary and initial conditions [112]. For applying the proposed strategy and the non-linear structure between natural gas price and weather temperature, we distinguishably appoint the second order differential equation and particularly the curvature formula to graphically model the non-linear relationship. Accordingly, we show the inverse relationship between natural gas price (P_c) and weather temperature (wt) by $\frac{dP_c}{dwt}<0$. The non-linearity would also be considered by $\frac{d^2P_c}{dwt^2}>0$. In aggregate, these two conditions of $\frac{dP_c}{dwt}<0$ and $\frac{d^2P_c}{dwt^2}>0$, really does provide the downward convexity of our proposed mathematical formula. Afterward, these mentioned derivational conditions are forced into the curvature equation (formula 4) to generally calculate natural gas price formula against weather temperature.

$$k=\frac{\left|\frac{d^2P_c}{dwt^2}\right|}{\left(1+\left[\frac{dP_c}{dwt}\right]^2\right)^{3/2}} \quad ,k>0 \tag{4}$$

4- 2- Scenario two

4- 2- 1- Strategies implied

The strategies devoted to scenario two are so similar to those appointed to scenario one. But distinguishably, we consider the following strategies separately to control the natural gas price.

1. COVARIANCE(weather temperature, natural gas price)<0.

2. COVARIANCE (demand, natural gas price)>0.

Hence in this strategy, It has been decided to reach to an explicit mathematical form of natural gas price= *f* (weather temperature, demand). In more detail, we are going to construct a non-linear formula which describes the inverse relationship between natural gas price and weather temperature while shows the direct relationship between natural gas price and demand simultaneously.

4- 2- 2- Methodology

To apply the proposed strategy and the non-linear structure between natural gas price (P_c), weather temperature (wt), and natural gas demand (D), we specifically adopt the second-order partial differential equation and particularly the Gaussian curvature formula to graphically model the non-linear relationship. Accordingly, we show the inverse relationship between natural gas price (P_c) and weather temperature (wt) by $\frac{\partial P_c(wt,D)}{\partial wt}<0$. Additionally, the direct relationship between natural gas price and demand is depicted by $\frac{\partial P_c(wt,D)}{\partial D}>0$. Furthermore, the equation's non-linearity would also be considered by $\frac{\partial^2 P_c(wt,D)}{\partial wt^2}.\frac{\partial^2 P_c(wt,D)}{\partial D^2}>0$. In aggregate, these three conditions of $\frac{\partial P_c(wt,D)}{\partial wt}<0$,$\frac{\partial P_c(wt,D)}{\partial D}>0$ and $\frac{\partial^2 P_c(wt,D)}{\partial wt^2}.\frac{\partial^2 P_c(wt,D)}{\partial D^2}>0$ really do provide the upward convexity of our proposed mathematical formula. Afterward, these mentioned derivational conditions are forced into the Gaussian curvature equation (formula 5) to generally calculate the natural gas price formula against weather temperature and demand.

$$k=\frac{\frac{\partial^2P_c(wt,D)}{\partial wt^2}.\frac{\partial^2P_c(wt,D)}{\partial D^2}-\left(\frac{\partial^2P_c(wt,D)}{\partial wt.\partial D}\right)^2}{\left(1+\left(\frac{\partial P_c(wt,D)}{\partial wt}\right)^2+\left(\frac{\partial P_c(wt,D)}{\partial D}\right)^2\right)^2} \quad ,k>0 \tag{5}$$

Having the aforementioned strategies accompanied to the mathematical equation of Gaussian curvature formula, we ultimately conclude the equation (6) shown below. We call it the *Gaussian- Curvature* strategic natural gas pricing (GES-NGP) formula from now on in this paper. (See Appendix 1)

$$k.\left[2\begin{pmatrix} t_1.\left[\frac{\partial U(t,\varepsilon)}{\partial \varepsilon}.D+c_2\right] \\ -s_1.\left[\frac{\partial U(t,\varepsilon)}{\partial t}.wt+c_1\right]\end{pmatrix}-s_1{}^2-t_1{}^2+1\right]^2 \tag{6}$$

$$+(F)^2-\lambda=0$$

$$,k,t_1,s_1>0\,,\lambda\gg F^2$$

5- Proposed mathematical models addressing to each strategies dedicated to two scenarios

5- 1- Proposed mathematical models relevant to scenario one

5- 1- 1- Basic mathematical form for scenario one

Before we start off this section, we need to make it clear that the differential equation structured in this paper would be categorized into the class of boundary value problems. Consequently, to accomplish the proposed mathematical model in this research and to solve the differential equation achieved by the curvature formula, first of all we need to have in hand the initial conditions of natural gas price against a predetermined weather temperature, together with the desired negative percent of slope of the price-weather temperature curve (namely the $\frac{dP_c}{dwt}$). For the present study, we deliberately restrict the negative percent of slope corresponding to each point of weather temperature to be lower than 0 and greater than -0.1. Besides, it should be indicated here that we consider the value of price, which is devoted to the upper limit of weather temperature (of a typical geographical region), as the initial value expressed by experts or policy makers to solve the differential equation. Having these conditions forcefully and certainly guarantees the declining form of the price-weather temperature form.

Ultimately, by applying the strategy and methodology which have been designated to making the differential equation (See Appendix 2), we simply obtain the following reference differential equation (7).

$$\frac{dP_c}{dwt} + \left(\left(\frac{\frac{d^2 P_c}{dwt^2}}{k} \right)^{2/3} - 1 \right)^{0.5} = 0 \tag{7}$$

The output of formula (7) would result the natural gas price (P_c) equation (formula 8) in terms of weather temperature (wt) and the scale value of curvature (k).

$$P_c = f(k, wt) \tag{8}$$

Until now, the natural gas price has been constructed using weather temperature. Continually, as stated before, we are to reconfigure formula (8) using the *income* variable. Thus, the following formula (9) is proposed to relate the *income* variable to curvature k.

$$k = f(income, \theta) = \frac{1}{\theta}\left[1 - \frac{1}{\ln(income)}\right], \quad \theta = \begin{cases} 100, & \alpha < income \\ 1000, & \beta < income \le \alpha \\ 10000, & income \le \beta \end{cases} \tag{9}$$

Where è is the correction coefficient, á would be the average income, which is considered as rather high income and β is also the average income which is seen as rather low income.

5- 1- 2- Modified mathematical form for scenario one:

In this point, we intend to modify formula (9) by substituting the variable *income* with its equivalent mathematical form, which is derived from the definition of income elasticity of demand. Income elasticity of demand is usually mathematically calculated by $E^D_{income} = \dfrac{d(D)/D}{d(income)/income}$.

Thus, formula (9) would be rewritten as shown in formula (10).

$$k = f(D, \theta) = \left| \frac{1}{\theta}\left[1 - \frac{1}{\ln\left(E^D_{income} . D . \, dincome/dD\right)}\right] \right|, \tag{10}$$

$$\theta = \begin{cases} 100, & \acute{\alpha} < D \\ 1000, & \acute{\beta} < D \le \acute{\alpha} \\ 10000, & D \le \acute{\beta} \end{cases}$$

Where θ is the correction coefficient; D stands for natural gas demand, $\alpha(in\, cubic\, meter)$ would be the natural gas demand which is considered as rather high demand, and $\beta(in\, cubic\, meter)$ is also the natural gas demand which is believed to be rather low demand. Meanwhile at this section, it is so worthy to mention that the income elasticity of demand (E^D_{income}) and the expression $dincome/dD$ (showing the first derivation of the income equation in terms of the demand factor) are assumed to be positive numbers for this study. To graphically illustrate, empirically verify, and support this assumption, we depict the graph of monthly average income-average natural gas demand data (Figure (5)). Data taken for 1980–2023 for Iran were used in the analysis. Using a curve-fitting tool (in MATLAB software), equation (11) with an R-squared of 0.875 is appropriately fitted to the data.

$$income = 2.319 \exp(0.003455D), \tag{11}$$

As it can be derived from formula (11), $d(income)/dD$ would be numerically positive almost everywhere and therefore, for positive values of natural gas demand and income, the E^D_{income} would also be a non-negative number as well.

5- 2- Proposed mathematical models relevant to scenario two

5- 2- 1- Basic form for scenario two

Using the *Gaussian-Curvature* strategic natural gas pricing (GES-NGP) formula (6) and solving it mathematically, we appropriately achieve the following basic parametric equation (12) relevant to the second scenario. (The proof is given in Appendix 1.)

$$P_c(wt, D) = \frac{1}{D.t_1}\left(t_1.c_2 - s_1.c_1 - \frac{\begin{pmatrix} t_1^2 + s_1^2 - 1 \\ + \left[\frac{\lambda - F^2}{k}\right]^{\frac{1}{2}} \end{pmatrix}}{2} \right) . \begin{pmatrix} \frac{D.t_1}{s_1}.wt + \left(\frac{D.t_1}{s_1}\right)^2.wt.\frac{1}{|a|.\sqrt{\pi}} \\ .\exp\left(-\left(\frac{D}{a}\right)^2\right).\ln(wt) \\ + D^4.\left(\frac{t_1}{s_1}\right)^3.wt.\frac{1}{|a|.a^2.\sqrt{\pi}} \\ .\exp\left(-\left(\frac{D}{a}\right)^2\right) + c_1.wt \end{pmatrix} + c_3, \tag{12}$$

Where $t_1, s_1, \lambda > 0$, a is a very small number close to zero.

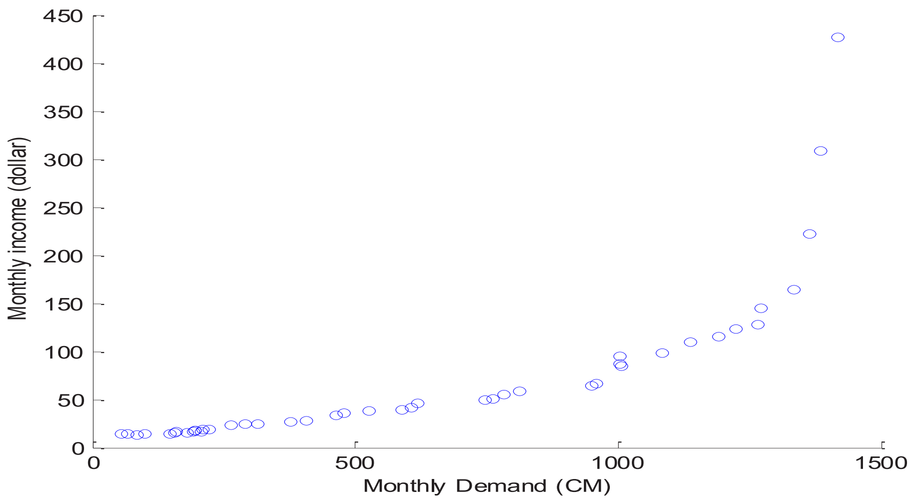

Fig. 5. Monthly income- natural gas demand graph.

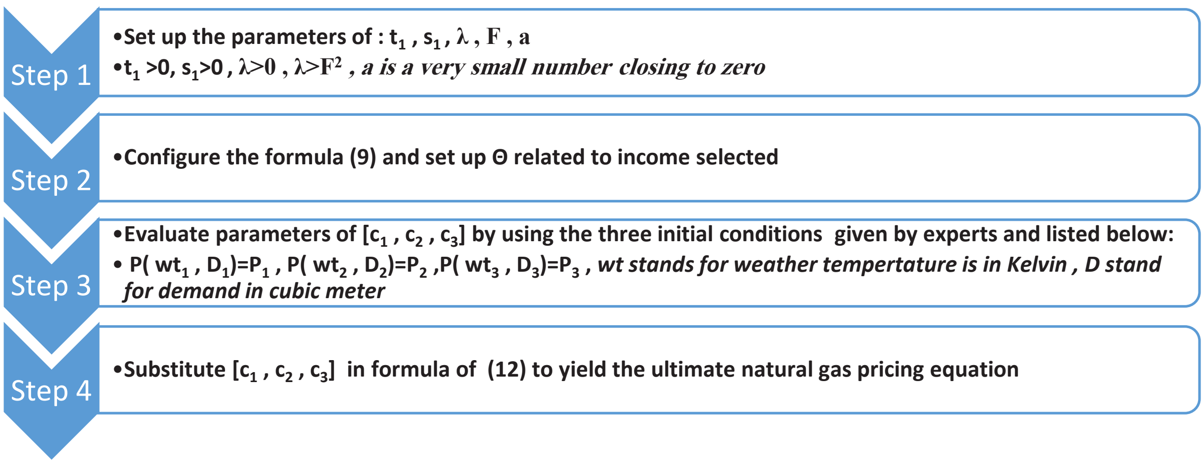

Fig. 6. The algorithm of using and configuring formula (12).

F would be free in sign and c_1, c_2, c_3 are constant numbers. Other symbols are previously defined in the context.

In this point, the algorithm corresponding to how to configure the parametric formula (12) to get the final equation of natural gas price ($P_c(wt, D)$) is depicted step by step in Figure 6.

6- Case study:

6- 1- Case study for scenario one:

6- 1- 1- Case study for based form:

In this section, a case study is given to depict the natural gas price-weather temperature curve along with presenting the non-linear formula. In this regard, we have the following initial conditions.

$\frac{dP_c}{dwt}$ =(D(P$_c$))(50°c) = -0.005,

P$_c$ (50°c) = 0.098 $\$/Giga\ joule$

So, by reference to the differential equation (formula 7) and using Maple software v.2016, we have the following equation.

$$Pc(wt) = \frac{1}{k}\left(\left(-50\,k - \frac{1}{40001}\sqrt{40001} + k\,wt - 1\right) \times \left(-50\,k - \frac{1}{40001}\sqrt{40001} + k\,wt + 1\right)\right) \tag{13}$$

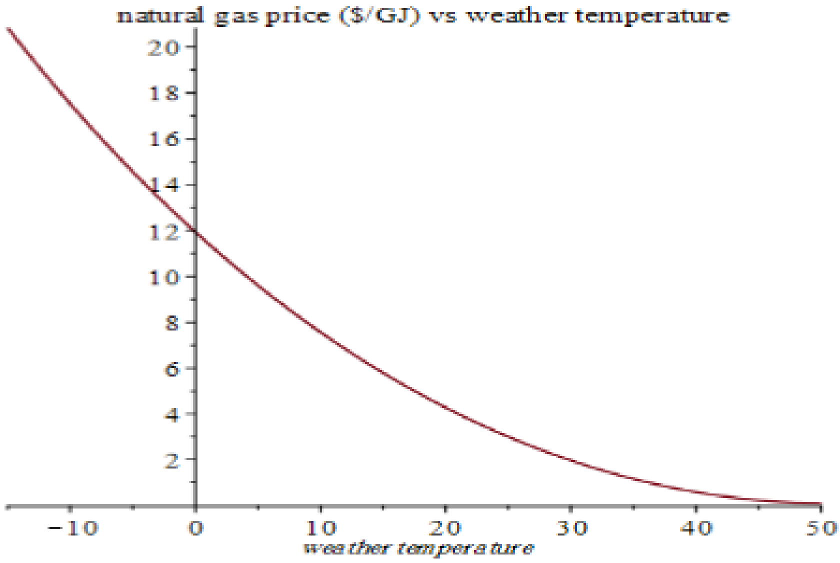

Fig. 7. Natural gas- weather temperature curve.

$$\times\sqrt{-\frac{1}{\left(-50\,k-\frac{1}{40001}\sqrt{40001}\right)^{2}+2\left(-50\,k-\frac{1}{40001}\sqrt{40001}\right)k\,wt+k^{2}\,wt^{2}-1}}\Bigg)$$

$$-\frac{1}{500}\,\frac{-49\,k-\frac{100000}{40001}\sqrt{40001}}{k}$$

In this step, we typically arrange the conditions associated with reference formula (9) as shown below to reconfigure equation (13).

$$\theta=\begin{cases}100, & 3000\,\$<income\\ 1000, & 1000\,\$<income\leq 3000\\ 10000, & income\leq 1000\,\$\end{cases},\qquad (14)$$

As an example, for income= 3500 $, we get reach to the k= 0.0087746 and the following curve related (figure (7)).

For income= 2000 $, we obtain the k= 0.00086844 and the following curve related (figure (8)).

At last, for income= 700 $, we obtain the k= 0.000084735 and the following related curve (Figure 9).

6- 1- 2- Case study for modified form

Having in hand the initial conditions considered for the previous case study and switching to formula (10) to calculate the curvature k, we set the following parameters of reference formula (10) as shown below.

$$\theta=\begin{cases}100, & 1500\,CM<D\\ 1000, & 500\,CM<D\leq 1500\;CM\\ 10000, & D\leq 500\;CM\end{cases},\qquad (15)$$

As an example, for D= 3000 CM, we get reach the k= 0.00919 and the following curve (Figure 10).

For D= 1000 CM, we obtain the k= 0.0007654 and the following curve related (figure (11)).

And finally, for D= 500 $, we obtain the k= 0.00004572 and the following related curve (Figure 12).

6- 2- Case study for scenario two:

In this section, it has been tried to test the mathematical model formulated in scenario two using the algorithm described already in Figure (6). Additionally, it should be noted that all the related mathematical calculations and outputs have been done in Maple software v.2016. Moreover, it is worthy to be mentioned that all the mandatory and configuration parameters are shown in the figures' captions.

6- 2- 1- Discussions for scenario two:

Considering the main constructed formula (12), we are now going to make some modified forms of it by substituting the parameters F, λ, and *a* with some mathematical expressions including fundamental variables of (wt, D). This is intentionally done to decrease the number of input configuring parameters to gain final natural gas price equation.

Thus for the first modification, we replace F with the expression of $\exp\left(-(wt.D)^{1/2}\right)$. Moreover, because of $\lambda>F^2$, It has been decided to have $\lambda=\alpha.\exp\left(-2.(wt.D)^{1/2}\right)$ for $\alpha>1$. So, there is a typical graph shown below for this modification.

For the second modification, we replace F with the expression of $\frac{1}{wt^2+D^3}$. Moreover, because of $\lambda>F^2$, It has been decided to have $\lambda=\alpha.\left(\frac{1}{wt^2+D^3}\right)^2$ for $\alpha>1$. So, there is a sample graph shown below for this modification.

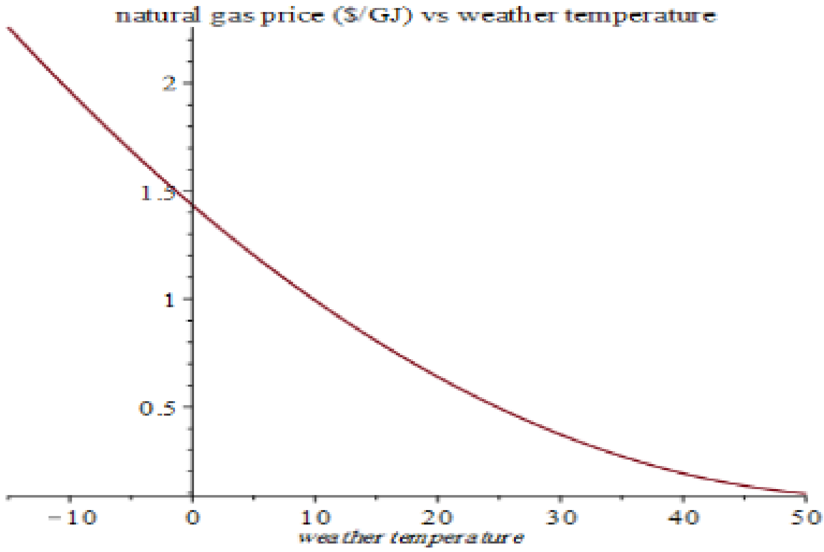

Fig. 8. Natural gas- weather temperature curve.

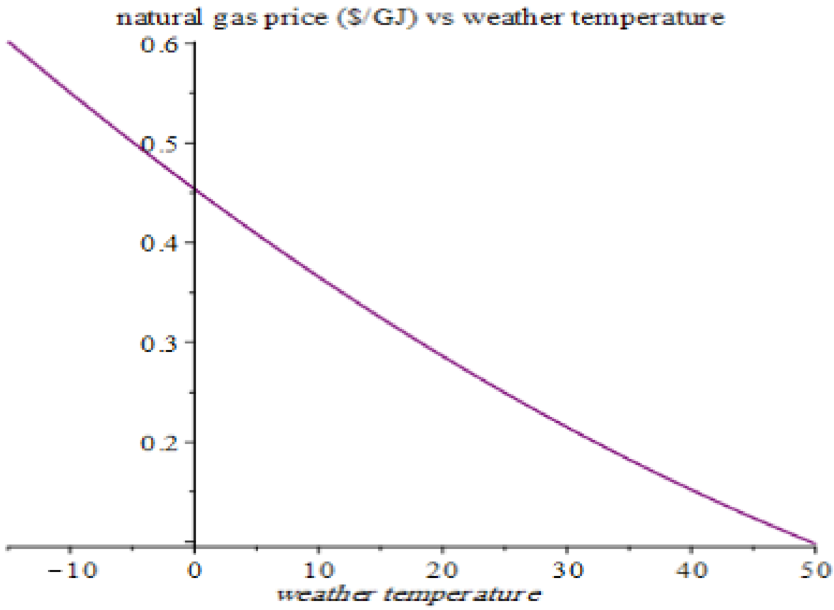

Fig. 9. Natural gas- weather temperature curve.

For the third modification, we replace F with the expression of $\exp\left(-(wt.D)^{1/2}\right)$ and also we substitute parameter *a* with the expression of $\frac{1}{wt.D}$. Moreover, because of $\lambda > F^2$, It has been decided to have $\lambda = \alpha.\exp\left(-2.(wt.D)^{1/2}\right)$ for $\alpha > 1$. So, there would be a typical graph shown below for this modification.

And now for the fourth modification, we are determined to examine formula (12) due to the change of variable *a* which inherently is a very small number close to zero. So, the terms of $\frac{1}{|a|.\sqrt{\pi}}.\exp\left(-\left(\frac{D}{a}\right)^2\right)$ and $\frac{1}{|a|.a^2.\sqrt{\pi}}.\exp\left(-\left(\frac{D}{a}\right)^2\right)$ would be very nearly limited to zero while the absolute value of *a* goes down rapidly (and not equally) to zero. This issue is clearly apparent in figures (22,23) below.

Consequently, formula (12) can be mathematically shortened to formula (16).

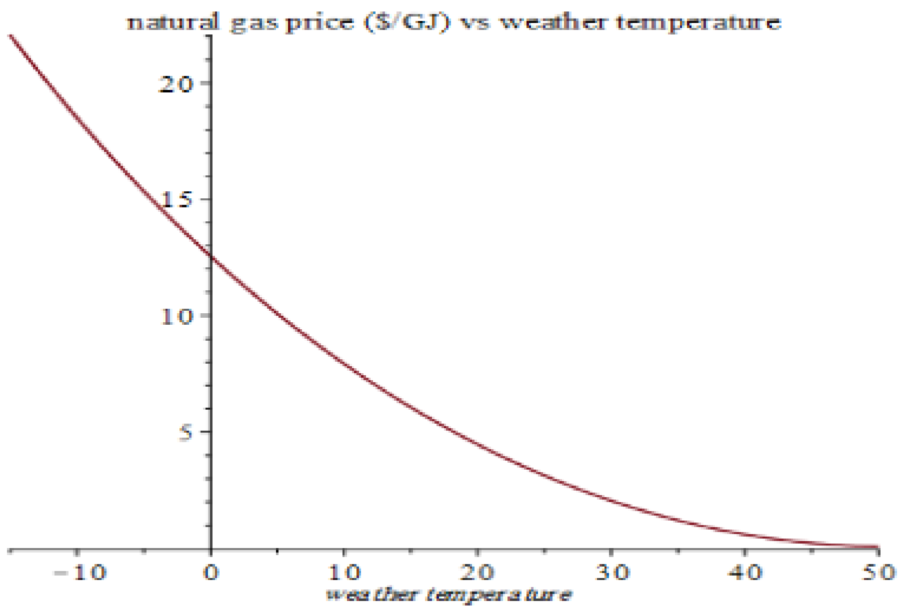

Fig. 10. Natural gas- weather temperature curve.

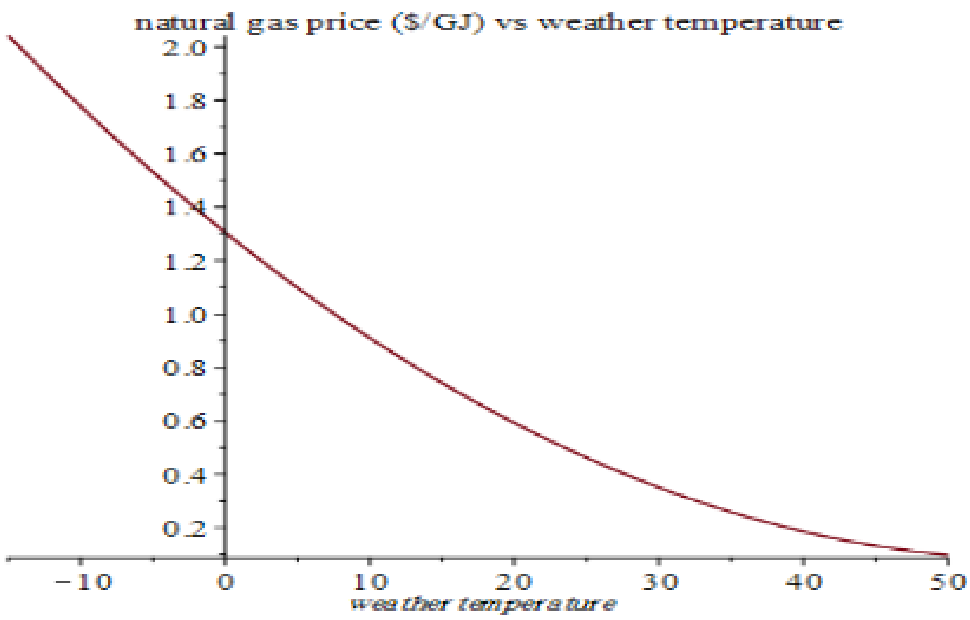

Fig. 11. Natural gas- weather temperature curve.

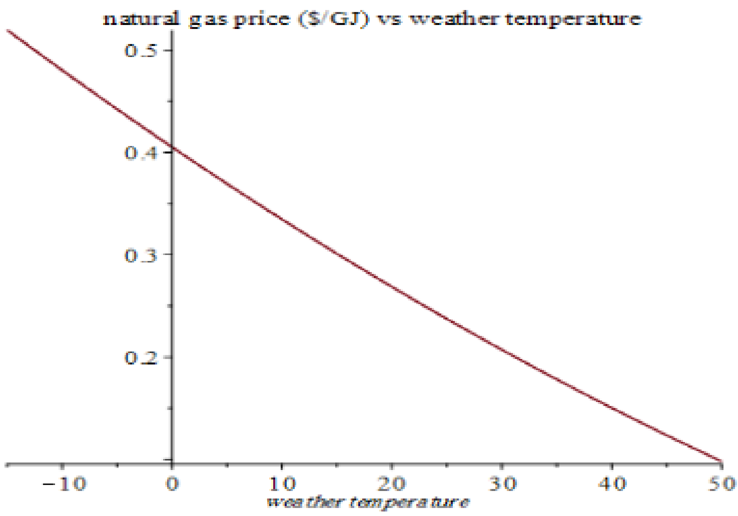

Fig. 12. Natural gas- weather temperature curve.

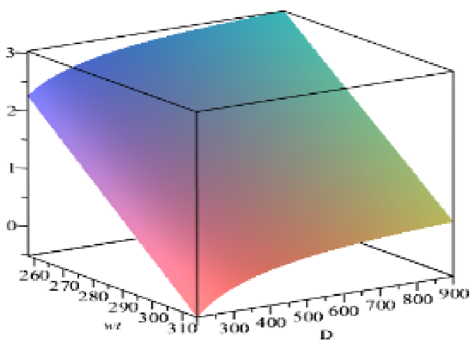

Fig. 13. Natural gas price graph for income = 4000 $ configured by t1=2, s1=1, λ=7.114, F=0.042 , a=0.0001, k= 0.0088, P1(258.15, 800)= 3 $, P2(273.15, 560)= 2.4 $, P3(283.15, 300)= 1.4 $.

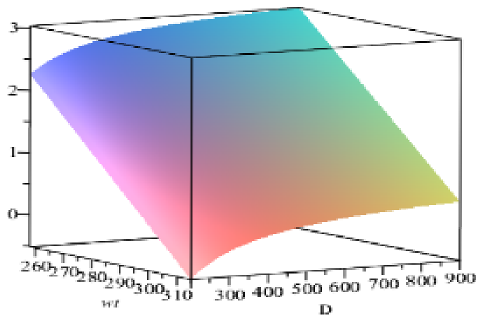

Fig. 14. Natural gas price graph for income = 4000 $ configured by t1=22, s1=55, λ=89, F=7 , a=0.001, k= 0.0088, P1(258.15, 800)= 3 $, P2(273.15, 560)= 2.2 $, P3(283.15, 300)= 1.4 $.

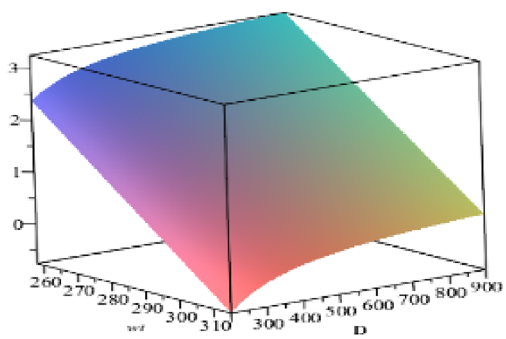

Fig. 15. Natural gas price graph for income = 4000 $ configured by t1=2, s1=1, λ=7.114, F=0.042 , a=0.0001, k= 0.0088, P1(258.15, 800)= 3 $, P2(273.15, 560)= 2.4 $, P3(283.15, 300)= 1.4 $.

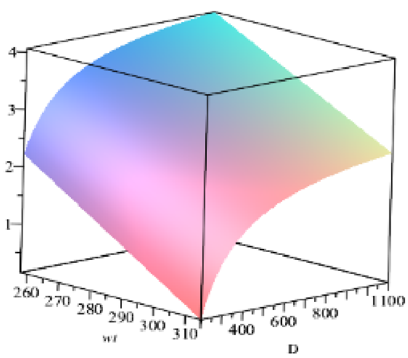

Fig. 16. Natural gas price graph for income = 700 $ configured by t1=2, s1=1, λ=7.114, F=2 , a=0.0001, k= 0.000084735, P1(258.15, 800)= 3.7 $, P2(273.15, 560)= 2.9 $, P3(283.15, 300)= 1.8 $.

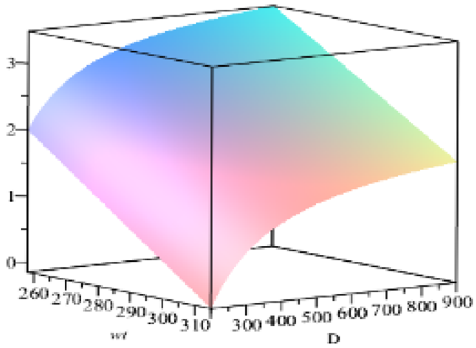

Fig. 17. Natural gas price graph for income = 700 $ configured by t1=2, s1=1, λ=7.114, F=2 , a=0.0001, k= 0.000084735, P1(265.15, 800)= 3.2 $, P2(283.15, 560)= 2.3 $, P3(293.15, 300)= 1.4 $.

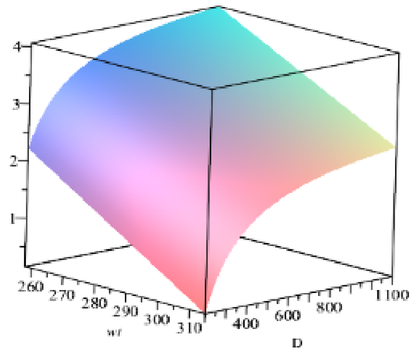

Fig. 18. Natural gas price graph for income = 3500 $ configured by t1=2, s1=1, λ=7.114, F=2 , a=0.0001, k= 0.00877, P1(265.15, 3500)= 2.8 $, P2(283.15, 3100)= 2.4 $, P3(293.15, 2700)= 2 $.

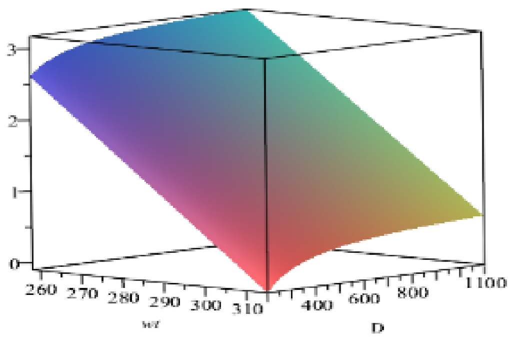

Fig. 19. Natural gas price graph for income = 3500 $ configured by t1=22, s1=51, a=0.00001,α=2, k= 0.00877, P1(265.15, 800)= 2.8 $, P2(283.15, 560)= 1.9 $, P3(293.15, 300)= 1.2 $.

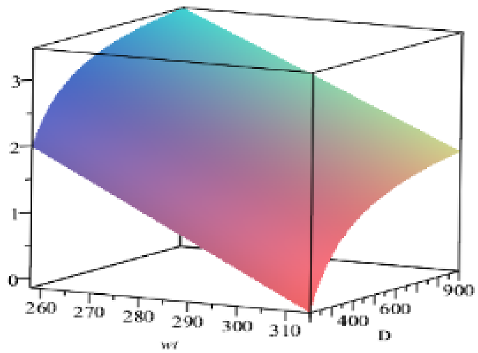

Fig. 20. Natural gas price graph for income = 3500 $ configured by t1=2, s1=5, a=0.0001,α=5, k= 0.00877, P1(265.15, 800)= 3.2 $, P2(283.15, 560)= 2.4 $, P3(293.15, 300)= 1.4 $.

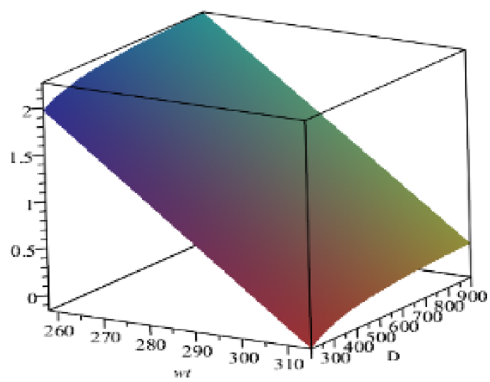

Fig. 21. Natural gas price graph for income = 3500 $ configured by t1=22, s1=51, α=2, k= 0.00877, P1(265.15, 800)= 2 $, P2(283.15, 560)= 1.3 $, P3(293.15, 300)= 0.8 $.

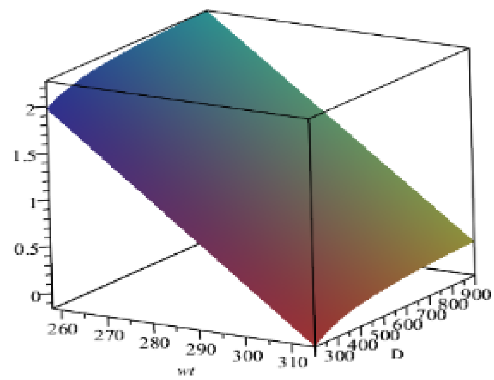

Fig. 21. Natural gas price graph for income = 3500 $ configured by t1=22, s1=51, α=2, k= 0.00877, P1(265.15, 800)= 2 $, P2(283.15, 560)= 1.3 $, P3(293.15, 300)= 0.8 $.

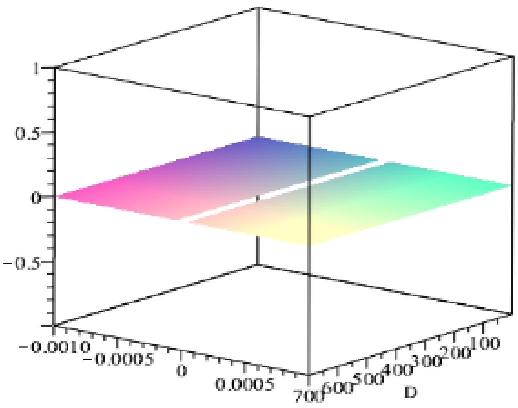

Fig. 22. plot3d(exp(-D²/a²)/abs(a), D = 300 .. 1100, a = -2 .. 2, axes = boxed).

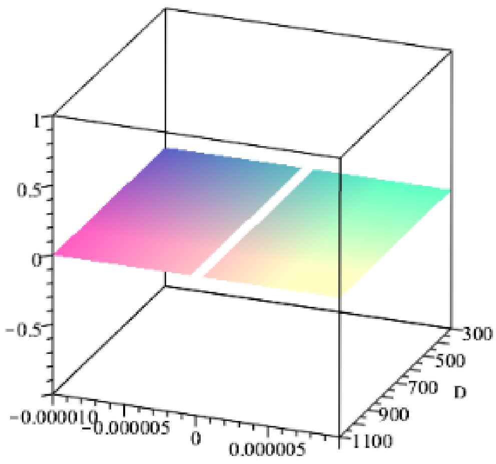

Fig. 23. plot3d(exp(-D²/a²)/(abs(a) . (a²)), D = 300 .. 1100, a = -0.1e-4 .. 0.1e-4, axes = boxed)

$$\lim_{a\to 0} P_c(wt, D) = \frac{1}{D.t_1}\left(t_1.c_2 - s_1.c_1 - \frac{\left(t_1^{\,2} + s_1^{\,2} - 1 + \left[\frac{\lambda - F^2}{k}\right]^{\frac{1}{2}}\right)}{2}\right) . \left(\begin{array}{l} \frac{D.t_1}{s_1}.wt + \left(\frac{D.t_1}{s_1}\right)^2 .wt.\frac{1}{|a|.\sqrt{\pi}} \\ .\exp\left(-\left(\frac{D}{a}\right)^2\right).\ln(wt) + D^4.\left(\frac{t_1}{s_1}\right)^3 \\ .wt.\frac{1}{|a|.a^2.\sqrt{\pi}}.\exp\left(-\left(\frac{D}{a}\right)^2\right) + c_1.wt \end{array}\right) + c_3 \;\; P_c(wt, D) \tag{16}$$

$$= \frac{1}{D.t_1}\left(t_1.c_2 - s_1.c_1 - \frac{\left(t_1^{\,2} + s_1^{\,2} - 1 + \left[\frac{\lambda - F^2}{k}\right]^{\frac{1}{2}}\right)}{2}\right) . \left(\frac{D.t_1}{s_1}.wt + c_1.wt\right) + c_3\,, \qquad \lambda \gg F^2\,,$$

At last the case study provided for formula (16) are graphically represented by figure (24).

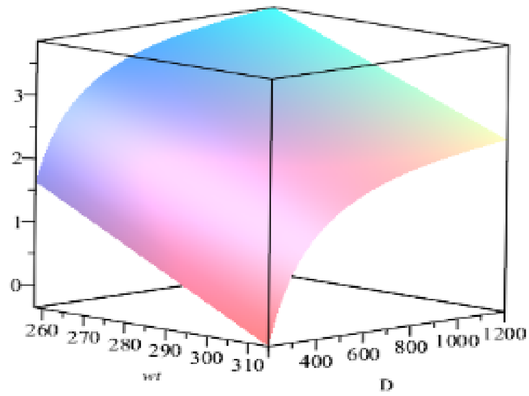

Fig. 24. Natural gas price graph for income = 4000 $ configured by t1=22, s1=31, λ=44555888, F=31 , k= 0.0088, P1(258.15, 900)= 3.7 $, P2(279.15, 590)= 2.8 $, P3(291.15, 380)= 1.9 $.

7- Sensitivity analysis with respect to *k* values:

In this section, we analyze the sensitivity of $P_c(wt, D)$ with respect to k using the first derivative approach and provide a thorough interpretation. Thus, for simplicity of calculations, at first we recall and rewrite the gas price formula as shown below.

$$P_c(wt,D) = \frac{1}{D.t_1}\left(t_1.c_2 - s_1.c_1 - \frac{\left(t_1^2 + s_1^2 - 1 + \left[\frac{\lambda - F^2}{k}\right]^{\frac{1}{2}} \right)}{2} \right) .G(wt,D) + c_3 \quad , \tag{17}$$

Where:

$$G(wt,D) = \left(\begin{array}{c} \frac{D.t_1}{s_1}.wt + \left(\frac{D.t_1}{s_1}\right)^2 \\ .wt.\frac{1}{|a|.\sqrt{\pi}}.\exp\left(-\left(\frac{D}{a}\right)^2\right) \\ .\ln(wt) + D^4.\left(\frac{t_1}{s_1}\right)^3 \\ .wt.\frac{1}{|a|.a^2.\sqrt{\pi}}.\exp\left(-\left(\frac{D}{a}\right)^2\right) \\ +c_1.wt \end{array} \right), \tag{18}$$

So only one term of $\left[\frac{\lambda - F^2}{k}\right]^{\frac{1}{2}}$ in (17) depend on k.

The derivative of $\left[\frac{\lambda - F^2}{k}\right]^{\frac{1}{2}}$ with respect to k is:

$$\frac{d\left[\frac{\lambda - F^2}{k}\right]^{\frac{1}{2}}}{dk} = -\frac{[\lambda - F^2]^{\frac{1}{2}}}{2k^{\frac{3}{2}}}, \tag{19}$$

Plugging equation (19) back into the derivative of $P_c(wt,D)$, we would have:

$$\frac{dP_c(wt,D)}{dk} = \frac{1}{D.t_1}.\frac{1}{2}.\frac{[\lambda - F^2]^{\frac{1}{2}}}{2k^{\frac{3}{2}}}.G(wt,D) = \frac{[\lambda - F^2]^{\frac{1}{2}}}{4D.t_1 k^{\frac{3}{2}}}.G(wt,D), \tag{20}$$

For $G(wt,D)$ which is always positive under normal operating conditions (positive demand and temperature) and also for conditions of $Dt_1 > 0$, $k > 0$ and $\lambda - F^2 > 0$, we have $\frac{dP_c(wt,D)}{dk} > 0$. This means that an increase in k leads to an increase in the gas price, but the relationship is strongly nonlinear due to the k^(-3/2) term.

Consequently, based on formula (20), some important mathematical interpretations are gained as follows:

1. Nonlinear amplification at small k

- The $k^{(-3/2)}$ factor implies that when k is small, the sensitivity is very high.
- Small changes in k under low-k conditions can lead to large changes in $P_c(wt,D)$.
- This is particularly important for risk management in markets with high volatility or uncertainty in k.

2. Diminishing effect for large k

- As k increases, the $k^{(-3/2)}$ factor decreases rapidly. This means price becomes less sensitive to changes in k as k grows. In other words, beyond a certain level of k, further improvements (or worsening) in k have smaller impacts on gas price.

3. Interaction with wt and D

- Sensitivity is proportional to $G(wt,D)$, which increases with both wt (temperature) and D (demand). Therefore, high demand or extreme temperature conditions amplify the impact of k on gas price.
- Economically, this reflects the fact that in periods of high demand or extreme weather, market adjustments (captured by k) have stronger effects on pricing.

4. Economic interpretation of k

- If k represents the natural gas market efficiency, adjustment speed, or a structural factor affecting supply-demand equilibrium, then:

Low k would result that market is more fragile, price reacts strongly.

High k would result that market is more stable, price reacts less.

So the derivative shows that uncertainty in k is critical for volatility assessment.

5. Risk management implication

- By analyzing $\frac{dP_c(wt,D)}{dk}$, we can identify critical regions of k where price sensitivity is extreme.
- Policy makers, traders, or risk managers can prioritize monitoring k and hedging strategies when k is low or moderate.

At this point, it has been decided to extend the sensitivity analysis to "relative sensitivity", which is also called elasticity of $P_c(wt,D)$ with respect to k. This measures the percentage change in $P_c(wt,D)$ for a 1% change in k. The relative sensitivity of $P_c(wt,D)$ with respect to k is defined as:

$$S_{P_c,k} = \frac{k}{P_c(wt,D)} \frac{dP_c(wt,D)}{dk}, \quad (21)$$

So, from our previous derivation given by $\frac{[\lambda - F^2]^{\frac{1}{2}}}{4D\,t_1 k^{\frac{3}{2}}}.G(wt,D)$, the relative sensitivity becomes as:

$$S_{P_c,k} = \frac{k}{P_c(wt,D)} \frac{[\lambda - F^2]^{\frac{1}{2}}}{4D.t_1 k^{\frac{3}{2}}}.G(wt,D) = \frac{[\lambda - F^2]^{\frac{1}{2}}}{4D.t_1 k^{\frac{1}{2}}}.\frac{G(wt,D)}{P_c(wt,D)}, \quad (22)$$

Therefore, based on formula (22), some significant mathematical interpretations are gained as follows:

1. Nonlinear inverse relationship with k
- The factor k^(-1/2) shows that relative sensitivity decreases with increasing k.
- Low k would result that small structural efficiency or high market friction, which then shows price is highly sensitive.
- High k would result in market stabilization, which then shows percentage change in price per unit change in k diminishes.

2. Dependence on $\frac{G(wt,D)}{P_c(wt,D)}$
- The relative effect also scales with $\frac{G(wt,D)}{P_c(wt,D)}$, which depends on temperature and demand.
- During periods of high demand or extreme temperature, the same percentage change in k leads to larger percentage changes in price.

3. Practical significance
- Relative sensitivity is more intuitive for economic decisions than absolute sensitivity. As an example: if $S_{P_c,k}$ = 0.2, a 10% increase in k leads to a 2% increase in price. This helps traders, policy makers, and analysts to quantify the impact of changes in structural or market parameters.

4. Risk and policy implications
- Monitoring k becomes critical when $S_{P_c,k}$ is high (low k and/or high (wt, D)) to avoid large unexpected price swings.
- Elasticity analysis allows dynamic hedging: if the market is in a sensitive regime, risk management strategies can be adjusted proportionally.

7- 1- Benchmark comparison with existing natural gas pricing models:

Although numerous mathematical models have been proposed for natural gas pricing and forecasting, their objectives and methodological structures differ considerably from the present study. Most existing approaches, including Vector Error Correction Models (VECM), Vector Autoregressive (VAR) models, Autoregressive Distributed Lag (ARDL), GARCH-family models, ARIMA models, and recent machine-learning and deep-learning techniques, have primarily been developed to forecast natural gas prices from historical market data or to investigate price volatility and market dynamics. These methods generally rely on time-series observations and emphasize statistical prediction accuracy. In contrast, the proposed Gaussian–*Curvature* Strategic Natural Gas Pricing (GES-NGP) model is not intended as a forecasting algorithm but rather as a strategic pricing framework for residential natural gas markets. The proposed model directly incorporates three fundamental demand-side determinants—weather temperature, residential natural gas demand, and consumers' average income—within a nonlinear curvature-based mathematical formulation. Unlike conventional econometric and machine-learning models, the proposed framework enables policymakers to explicitly control pricing behavior through configurable mathematical parameters while preserving the nonlinear relationship among the governing variables. Another important distinction is the flexibility of the proposed curvature formulation. The curvature coefficient can be reconfigured using additional socioeconomic or market-related variables, allowing the pricing model to be extended for different regulatory environments and policy objectives. Consequently, the proposed framework provides a mathematically transparent and adaptable pricing mechanism rather than solely improving statistical forecasting performance.

Overall, compared with existing pricing and forecasting models, the proposed GES-NGP framework offers four principal contributions:

simultaneous integration of weather temperature, residential demand, and consumers' average income within a unified nonlinear pricing equation;

a curvature-based mathematical structure capable of representing nonlinear pricing behavior;

flexible incorporation of additional policy or market variables through parameter reconfiguration; and

a strategic pricing methodology specifically designed for residential tariff design rather than short-term market price prediction.

These characteristics distinguish the proposed model from existing econometric models (see Table 2) and demonstrate its potential value as a complementary decision-support tool for natural gas pricing policy.

8- Conclusion and Future Research:

In this research, we have presented a mathematical model for natural gas pricing designed for the regional residential sector. Daily weather temperature, together with final demand, represents one of the fundamental uncertainties associated with determining natural gas prices by firms and gas companies around the world. Hence, weather temperature

Table 2. The main advantages of the proposed model considering other existing models.

Method	Main Objective	Main Variables	Main Limitation	Advantage of Proposed GES-NGP
VECM	Long-run price relationship	Oil price, gas price	Historical data dependent	Incorporates policy-based nonlinear pricing
VAR	Market interaction analysis	Temperature, storage, supply	Mainly forecasting	Direct strategic pricing formulation
GARCH	Price volatility modelling	Historical prices, weather	Focuses on volatility only	Explicit pricing equation
ARIMA	Time-series forecasting	Historical prices	Linear statistical approach	Nonlinear curvature formulation
LSBoost / ANN / Deep Learning	Price prediction	Large datasets	Low interpretability	Transparent mathematical model
Proposed GES-NGP	**Strategic residential pricing**	**Temperature, demand, income**	**Configurable mathematical framework**	**Flexible, interpretable, policy-oriented pricing model**

and demand, as the two most important factors affecting the natural gas market, have been selected to formulate the proposed pricing models.

Due to the nonlinear nature of natural gas pricing behavior with respect to weather temperature and natural gas demand, differential equations have been employed as a flexible mathematical approach to construct the pricing model in terms of weather temperature and natural gas demand under two different scenarios. Therefore, a curvature-based formula has been adopted in the present study. By using the curvature formula as a mathematical approach, natural gas prices can be strategically managed for different levels of income and demand simultaneously under two predefined scenarios.

Moreover, it should be noted that the main assumption of this study is that charging higher natural gas prices when weather temperature decreases can, to some extent, control natural gas consumption in the residential sector. The proposed approach can be flexibly utilized by policymakers and strategists who are interested in adjusting natural gas prices while considering weather temperature, residential end users' average income, and demand simultaneously.

The fundamental contribution of the present study lies in the use of the curvature formula, which provides the flexibility to incorporate different applicable strategies into the pricing mechanism. More importantly, the main innovation of this work is that the natural gas pricing model has been reconfigured in such an adaptable manner that policymakers can incorporate other influential parameters, such as the natural gas market risk parameter (k), to derive alternative mathematical pricing equations.

For future research, it is suggested to conduct similar investigations by considering weather temperature together with other influential variables, such as GDP (Gross Domestic Product), taxation, and the prices of other energy sources. Furthermore, the mathematical methodology employed in this study is based on deterministic values. Therefore, future studies may consider fuzzy numbers as an alternative computational approach.

Additionally, for the natural gas pricing model presented in Scenario Two, three mathematical suggestions are proposed for future investigations.

1. For $u(t,\varepsilon)$ calculate $\dfrac{\partial u(t,\varepsilon)}{\partial \varepsilon}$ then apply it to the equation of $\dfrac{\partial P_c(wt,D)}{\partial D}=\dfrac{\partial U(t,\varepsilon)}{\partial \varepsilon}.D+c_2$ to obtain the final equation of P_c(wt, D). (See formula (33) in Appendix 1)

2. By the Legendre assumption of

$$u(t,\varepsilon)=wt.t+D.\varepsilon-P_c(wt,D),\ \frac{\partial P_c(wt,D)}{\partial wt}=t\ ,\frac{\partial P_c(wt,D)}{\partial D}=\varepsilon$$

solve equation (11) in the Appendix.

3. Suppose the following mathematical forms of parameter F for more discussions.

a) $F=\left(wt^2+D^2+wt.D\right)^{-1}$, b) $F=\exp\{-\left(wt^2+D^2+wt.D\right)^{1/2}\}$,

c) $F=\dfrac{1}{\left(wt^2+D^2+wt.D\right)^{1/2}}$

Funding: There is no funding to support this work.

Conflict of Interest: There is no conflict of interest.

Availability of data and material: All data used from other sources are referenced.

Code availability: Not applicable

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Appendix 1:

The mathematical proof of formula (12) is given below step by step.

Consider the Gaussian curvature formula as follows:

$$k = \frac{\frac{\partial^2 P_c(wt, D)}{\partial wt^2} . \frac{\partial^2 P_c(wt, D)}{\partial D^2} - \left(\frac{\partial^2 P_c(wt, D)}{\partial wt . \partial D}\right)^2}{\left(1 + \left(\frac{\partial P_c(wt, D)}{\partial wt}\right)^2 + \left(\frac{\partial P_c(wt, D)}{\partial D}\right)^2\right)^2} \quad , k > 0 \quad , (1)$$

So the Gaussian curvature k can be rewritten as shown below:

$$k . \left(1 + \left(\frac{\partial P_c(wt, D)}{\partial wt}\right)^2 + \left(\frac{\partial P_c(wt, D)}{\partial D}\right)^2\right)^2 = \frac{\partial^2 P_c(wt, D)}{\partial wt^2} . \frac{\partial^2 P_c(wt, D)}{\partial D^2} - \left(\frac{\partial^2 P_c(wt, D)}{\partial wt . \partial D}\right)^2 , (2)$$

Thus:

$$\sqrt{k} . \left(1 + \left(\frac{\partial P_c(wt, D)}{\partial wt}\right)^2 + \left(\frac{\partial P_c(wt, D)}{\partial D}\right)^2\right) = \begin{cases} + \left(\frac{\partial^2 P_c(wt, D)}{\partial wt^2} . \frac{\partial^2 P_c(wt, D)}{\partial D^2} - \left(\frac{\partial^2 P_c(wt, D)}{\partial wt . \partial D}\right)^2\right)^{1/2} \\ - \left(\frac{\partial^2 P_c(wt, D)}{\partial wt^2} . \frac{\partial^2 P_c(wt, D)}{\partial D^2} - \left(\frac{\partial^2 P_c(wt, D)}{\partial wt . \partial D}\right)^2\right)^{1/2} \end{cases} , (3)$$

We deliberately choose the positive form on the right-hand side of the above equation for further operations.

$$\sqrt{k} . \left(1 + \left(\frac{\partial P_c(wt, D)}{\partial wt}\right)^2 + \left(\frac{\partial P_c(wt, D)}{\partial D}\right)^2\right) = + \left(\frac{\partial^2 P_c(wt, D)}{\partial wt^2} . \frac{\partial^2 P_c(wt, D)}{\partial D^2} - \left(\frac{\partial^2 P_c(wt, D)}{\partial wt . \partial D}\right)^2\right)^{1/2} \quad , \quad (4)$$

For the supposed strategies taken for scenario two, we can rewrite them as follows:

$$\frac{\partial P_c(wt, D)}{\partial wt} < 0 \rightarrow \frac{\partial P_c(wt, D)}{\partial wt} + s_1 = 0 \ , \qquad s_1 > 0 \ , \qquad (5)$$

$$\frac{\partial P_c(wt, D)}{\partial D} > 0 \rightarrow \frac{\partial P_c(wt, D)}{\partial wt} - t_1 = 0 \ , \qquad t_1 > 0 \ , \ (6)$$

Now , these two strategies would be mathematically integrated same as below:

$$\left(\frac{\partial P_c(wt, D)}{\partial wt} + s_1\right)^2 = 0 \ \rightarrow \left(\frac{\partial P_c(wt, D)}{\partial wt}\right)^2 + 2 . s_1 . \frac{\partial P_c(wt, D)}{\partial wt} + {s_1}^2 = 0 \ , (7)$$

$$\left(\frac{\partial P_c(\,wt,D)}{\partial D}-t_1\right)^2=0\ \rightarrow\left(\frac{\partial P_c(\,wt,D)}{\partial D}\right)^2-2.t_1.\frac{\partial P_c(\,wt,D)}{\partial D}+{t_1}^2=0\ ,\quad(8)$$

Consequently:

$$\left(\frac{\partial P_c(\,wt,D)}{\partial wt}\right)^2+2.s_1.\frac{\partial P_c(\,wt,D)}{\partial wt}+{s_1}^2+\left(\frac{\partial P_c(\,wt,D)}{\partial D}\right)^2-2.t_1.\frac{\partial P_c(\,wt,D)}{\partial D}+{t_1}^2=0\ ,(9)$$

By adding 1 to each side of equation (9) and rewriting it, we would have:

$$1+\left(\frac{\partial P_c(\,wt,D)}{\partial wt}\right)^2+\left(\frac{\partial P_c(\,wt,D)}{\partial D}\right)^2=1-2.s_1.\frac{\partial P_c(\,wt,D)}{\partial wt}+2.t_1.\frac{\partial P_c(\,wt,D)}{\partial D}-{s_1}^2-{t_1}^2\,,(10)$$

Substituting the right-hand side of equation (10) in (4), It will result in:

$$\frac{1}{\sqrt{k}}.\left(\frac{\partial^2 P_c(\,wt,D)}{\partial wt^2}.\frac{\partial^2 P_c(\,wt,D)}{\partial D^2}-\left(\frac{\partial^2 P_c(\,wt,D)}{\partial wt.\partial D}\right)^2\right)^{1/2}+\ 2\left(s_1.\frac{\partial P_c(\,wt,D)}{\partial wt}-t_1.\frac{\partial P_c(\,wt,D)}{\partial D}\right)$$
$$+{s_1}^2+{t_1}^2-1=0\,,\qquad(11)$$

Equation (11) is a non-linear second-order partial differential equation, which is typically a form of the generally known Monge-Ampere equation. At this point, it will be tried to solve it using the assumptions listed below:

$$For\ \frac{\partial^2 P_c(\,wt,D)}{\partial wt^2}and\ \ \frac{\partial^2 P_c(\,wt,D)}{\partial D^2}\,,there\ would\ be\ two\ first\ order\ partail\ differential\ equations\ like\ \frac{\partial U(t,\varepsilon)}{\partial t}\ and\ \frac{\partial U(t,\varepsilon)}{\partial \varepsilon}\ respectively\ so\ that:$$

$$\frac{\partial^2 P_c(\,wt,D)}{\partial wt^2}=\frac{\partial U(t,\varepsilon)}{\partial t}\ ,\qquad\qquad \frac{\partial^2 P_c(\,wt,D)}{\partial D^2}=\frac{\partial U(t,\varepsilon)}{\partial \varepsilon}\ ,\qquad(12)$$

And also we assume that:

$$\frac{\partial^2 P_c(\,wt,D)}{\partial wt.\partial D}=F\ \ ,\qquad(13)$$

Thus, by integrating equation (12), we would have:

$$\frac{\partial P_c(\,wt,D)}{\partial wt}=\frac{\partial U(t,\varepsilon)}{\partial t}.wt+c_1\,,\qquad \frac{\partial P_c(\,wt,D)}{\partial D}=\frac{\partial U(t,\varepsilon)}{\partial \varepsilon}.D+c_2\ \ ,\qquad(14)$$

$$\frac{\partial^2 P_c(\,wt,D)}{\partial wt^2}.\frac{\partial^2 P_c(\,wt,D)}{\partial D^2}$$
$$=k.\left[2\left(t_1.\left[\frac{\partial U(t,\varepsilon)}{\partial \varepsilon}.D+c_2\right]-s_1.\left[\frac{\partial U(t,\varepsilon)}{\partial t}.wt+c_1\right]\right)-{s_1}^2-{t_1}^2+1\right]^2$$
$$+(F)^2\,,\qquad(15)$$

For the strategy being assumed as $\frac{\partial^2 P_c(wt,D)}{\partial wt^2} . \frac{\partial^2 P_c(wt,D)}{\partial D^2} > 0$, we therefore can rewrite it to be shown as an equation as follows:

$$\frac{\partial^2 P_c(wt,D)}{\partial wt^2} . \frac{\partial^2 P_c(wt,D)}{\partial D^2} - \lambda = 0 \,, \qquad \lambda > 0 \,, \qquad (16)$$

Considering equations (15) and (16), it clearly results in equation (17), which we name the *Gaussian-Curvature* strategic pricing formula:

$$k.\left[2\left(t_1.\left[\frac{\partial U(t,\varepsilon)}{\partial \varepsilon}.D + c_2\right] - s_1.\left[\frac{\partial U(t,\varepsilon)}{\partial t}.wt + c_1\right]\right) - {s_1}^2 - {t_1}^2 + 1\right]^2 + (F)^2 - \lambda = 0 \;, (17)$$

As a consequence:

$$2\left(t_1.\left[\frac{\partial U(t,\varepsilon)}{\partial \varepsilon}.D + c_2\right] - s_1.\left[\frac{\partial U(t,\varepsilon)}{\partial t}.wt + c_1\right]\right) - {s_1}^2 - {t_1}^2 + 1 = \begin{cases} +\left[\dfrac{\lambda - F^2}{k}\right]^{1/2} \\ -\left[\dfrac{\lambda - F^2}{k}\right]^{1/2} \end{cases} , \qquad (18)$$

We pick up the positive form of the right-hand side of equation (18) and continue for further mathematical calculations as follows.

By assuming $t \equiv x$, $\varepsilon \equiv t$ and rewriting equation (18), we obtain:

$$t_1.\left[\frac{\partial U(x,t)}{\partial t}.D + c_2\right] - s_1.\left[\frac{\partial U(x,t)}{\partial x}.wt + c_1\right] = \frac{{s_1}^2 + {t_1}^2 - 1 + \left[\frac{\lambda - F^2}{k}\right]^{1/2}}{2} \quad , \qquad (19)$$

Using the Laplace transformation and its mathematical characteristics, we would get access to:

$$D.t_1.L\big(U_t(x,t)\big) + L(t_1.c_2) - s_1.wt.L\big(U_x(x,t)\big) - L(s_1.c_1) = L\left[\frac{{s_1}^2 + {t_1}^2 - 1 + \left[\frac{\lambda - F^2}{k}\right]^{1/2}}{2}\right], (20)$$

Therefore, we can reach to:

$$D.t_1.\big(S.U(x,S) - U(x,0)\big) + L(t_1.c_2) - s_1.wt.\frac{dU(x,S)}{dx} - L(s_1.c_1) = L\left[\frac{{s_1}^2 + {t_1}^2 - 1 + \left[\frac{\lambda - F^2}{k}\right]^{1/2}}{2}\right], \qquad (21)$$

We assume the initial condition of $U(x,0) = 0$ and x>0, so equation (21) is rewritten as shown below.

$$\frac{dU(x,S)}{dx} - \frac{D.t_1}{s_1.wt}.\left(S.U(x,S)\right) = \frac{L(t_1.c_2)}{s_1.wt} - \frac{L(s_1.c_1)}{s_1.wt} - \frac{L\left[\frac{{s_1}^2 + {t_1}^2 - 1 + \left[\frac{\lambda - F^2}{k}\right]^{1/2}}{2}\right]}{s_1.wt} \quad , \qquad (22)$$

So we solve it by finding the integrating factor.

$$the\ integrating\ factor = \mu = e^{\int -\frac{D.t_1}{s_1.wt}.S.dx} = e^{-\frac{D.t_1}{s_1.wt}.S.x} \quad , \qquad (23)$$

Thus we have:

$$\frac{d\left[e^{-\frac{D.t_1}{s_1.wt}.S.x}.U(x,S)\right]}{dx} = e^{-\frac{D.t_1}{s_1.wt}.S.x}.\left[\frac{L(t_1.c_2)}{s_1.wt} - \frac{L(s_1.c_1)}{s_1.wt} - \frac{L\left[\frac{{s_1}^2 + {t_1}^2 - 1 + \left[\frac{\lambda - F^2}{k}\right]^{1/2}}{2}\right]}{s_1.wt}\right] \quad , \qquad (24)$$

We integrate both sides of (24) to get:

$$e^{-\frac{D.t_1}{s_1.wt}.S.x}.U(x,S) = \left[\frac{L(t_1.c_2)}{s_1.wt} - \frac{L(s_1.c_1)}{s_1.wt} - \frac{L\left[\frac{{s_1}^2 + {t_1}^2 - 1 + \left[\frac{\lambda - F^2}{k}\right]^{1/2}}{2}\right]}{s_1.wt}\right].\left(-\frac{s_1.wt}{D.t_1.S}e^{-\frac{D.t_1}{s_1.wt}.S.x} + z\right) \quad ,$$

$$z\ is\ a\ constant \quad , \qquad (25)$$

Consequently we would have:

$$U(x,S) = \left[\frac{L(t_1.c_2)}{s_1.wt} - \frac{L(s_1.c_1)}{s_1.wt} - \frac{L\left[\frac{{s_1}^2 + {t_1}^2 - 1 + \left[\frac{\lambda - F^2}{k}\right]^{1/2}}{2}\right]}{s_1.wt}\right].\left(-\frac{s_1.wt}{D.t_1.s} + z.e^{\frac{D.t_1}{s_1.wt}.S.x}\right) \quad , \qquad (26)$$

With boundary and initial condition of $U(0,S) = 0$ and S>0 , we really have:

$$z = \frac{s_1.wt}{D.t_1.S} \quad , \qquad (27)$$

Then $U(x,S)$ in (26) can be rewritten as below:

$$U(x,S)=\left[\frac{L(t_1.c_2)}{s_1.wt}-\frac{L(s_1.c_1)}{s_1.wt}-\frac{L\left[\frac{{s_1}^2+{t_1}^2-1+\left[\frac{\lambda-F^2}{k}\right]^{1/2}}{2}\right]}{s_1.wt}\right].\left(\frac{s_1.wt}{D.t_1}\right).\left[\frac{1}{S}.e^{\frac{D.t_1}{s_1.wt}.S.x}-\frac{1}{S}\right] ,(28)$$

By applying inverse Laplace we conclude:

$$u(x,t)=\frac{1}{D.t_1}.\left[t_1.c_2-s_1.c_1-\frac{\left({s_1}^2+{t_1}^2-1+\left[\frac{\lambda-F^2}{k}\right]^{1/2}\right)}{2}\right].L^{-1}\left[\frac{1}{S^2}.e^{\frac{D.t_1}{s_1.wt}.S.x}-\frac{1}{S^2}\right], \quad (29)$$

For the reason of mathematical calculating complexion related to the inverse Laplace of function of $\frac{1}{S^2}.e^{\frac{D.t_1}{s_1.wt}.S.x}$, now we substitute the expression of $e^{\frac{D.t_1}{s_1.wt}.S.x}$ by its equivalent series-like function (We do have $e^X=\sum_0^\infty\frac{X^n}{n!}$).

$$u(x,t)=\frac{1}{D.t_1}.\left[t_1.c_2-s_1.c_1-\frac{\left({s_1}^2+{t_1}^2-1+\left[\frac{\lambda-F^2}{k}\right]^{1/2}\right)}{2}\right].L^{-1}\left[\frac{1}{S^2}.\sum_0^\infty\frac{\left(\frac{D.t_1}{s_1.wt}.S.x\right)^n}{n!}-\frac{1}{S^2}\right], \quad (30)$$

Due to the presence of $\sum$ in $u(x,t)$, we have to approximate $\sum$ for some limited number of n. Therefore, we estimate it here for n=0 to 3 for simplicity. Hence, we would have:

$$u(x,t)=\frac{1}{D.t_1}.\left[t_1.c_2-s_1.c_1-\frac{\left({s_1}^2+{t_1}^2-1+\left[\frac{\lambda-F^2}{k}\right]^{1/2}\right)}{2}\right].L^{-1}\left[\frac{D.t_1}{s_1.wt}.\frac{1}{S}.x+\frac{1}{2}.\left(\frac{D.t_1.x}{s_1.wt}\right)^2+\frac{1}{6}.\left(\frac{D.t_1.x}{s_1.wt}\right)^3.S\right], \quad (31)$$

Then applying inverse Laplace:

$$u(x,t)=\frac{1}{D.t_1}.\left[t_1.c_2-s_1.c_1-\frac{\left({s_1}^2+{t_1}^2-1+\left[\frac{\lambda-F^2}{k}\right]^{1/2}\right)}{2}\right].\left[\frac{D.t_1}{s_1.wt}.x+\frac{1}{2}.\left(\frac{D.t_1.x}{s_1.wt}\right)^2.Dirac(t)+\frac{1}{6}.\left(\frac{D.t_1.x}{s_1.wt}\right)^3.Dirac(1,t)\right]\ ,\qquad(32)$$

Where $Dirac$ refers to the known function of the *Dirac delta function*.

Substituting again $x\equiv t\ ,\ \ t\equiv\varepsilon$:

$$u(t,\varepsilon)=\frac{1}{D.t_1}.\left[t_1.c_2-s_1.c_1-\frac{\left({s_1}^2+{t_1}^2-1+\left[\frac{\lambda-F^2}{k}\right]^{1/2}\right)}{2}\right].\left[\frac{D.t_1}{s_1.wt}.t+\frac{1}{2}.\left(\frac{D.t_1.t}{s_1.wt}\right)^2.Dirac(\varepsilon)+\frac{1}{6}.\left(\frac{D.t_1.t}{s_1.wt}\right)^3.Dirac(1,\varepsilon)\right]\ ,\qquad(33)$$

Now we form the derivation of $u(t,\varepsilon)$ to variable t.

$$\frac{\partial u(t,\varepsilon)}{\partial t}=\frac{1}{D.t_1}.\left[t_1.c_2-s_1.c_1-\frac{\left({s_1}^2+{t_1}^2-1+\left[\frac{\lambda-F^2}{k}\right]^{1/2}\right)}{2}\right].\left[\frac{D.t_1}{s_1.wt}+\left(\frac{D.t_1}{s_1.wt}\right)^2.t.Dirac(\varepsilon)+\frac{1}{2}.\left(\frac{D.t_1}{s_1.wt}\right)^3.t^2.Dirac(1,\varepsilon)\right]\ ,\qquad(34)$$

As previously assumed in (14), we have $\frac{\partial P_c(\,wt,D)}{\partial wt}=\frac{\partial U(t,\varepsilon)}{\partial t}.wt+c_1$, so in this step we get:

$$\frac{\partial P_c(\,wt,D)}{\partial wt}=\left(\frac{1}{D.t_1}.\left[t_1.c_2-s_1.c_1-\frac{\left({s_1}^2+{t_1}^2-1+\left[\frac{\lambda-F^2}{k}\right]^{1/2}\right)}{2}\right].\left[\frac{D.t_1}{s_1.wt}+\left(\frac{D.t_1}{s_1.wt}\right)^2.t.Dirac(\varepsilon)+\frac{1}{2}.\left(\frac{D.t_1}{s_1.wt}\right)^3.t^2.Dirac(1,\varepsilon)\right]\right).wt+c_1\ ,\qquad(35)$$

We integrate both sides of (35) to get:

$$P_c(wt, D) = \frac{1}{D.t_1}.\left[t_1.c_2 - s_1.c_1 - \frac{\left({s_1}^2 + {t_1}^2 - 1 + \left[\frac{\lambda - F^2}{k}\right]^{\frac{1}{2}}\right)}{2}\right].\left[\frac{D.t_1}{s_1}.wt + \left(\frac{D.t_1}{s_1}\right)^2.t.Dirac(\varepsilon).\ln(wt) - \frac{1}{2}.\left(\frac{D.t_1}{s_1}\right)^3.t^2.Dirac(1,\varepsilon).\frac{1}{wt} + c_1.wt\right] + c_3\,, \quad (36)$$

By assumption of $t \equiv wt$, $\varepsilon \equiv D$ and also the Gaussian estimation of the *Dirac delta function* shown below:

$$Dirac(\varepsilon) = \lim_{a\to 0} \delta(\varepsilon) = \frac{1}{|a|\sqrt{\pi}} e^{-\left(\frac{\varepsilon}{a}\right)^2}, Dirac(1,\varepsilon) = \frac{d\big(Dirac(\varepsilon)\big)}{d\varepsilon} = -\frac{2\varepsilon}{a^2|a|\sqrt{\pi}} e^{-\left(\frac{\varepsilon}{a}\right)^2}, \quad a \to 0\,, \quad (37)$$

Then we ultimately conclude that:

$$P_c(wt, D) = \frac{1}{D.t_1}.\left[t_1.c_2 - s_1.c_1 - \frac{\left({s_1}^2 + {t_1}^2 - 1 + \left[\frac{\lambda - F^2}{k}\right]^{\frac{1}{2}}\right)}{2}\right].\left[\frac{D.t_1}{s_1}.wt + \left(\frac{D.t_1}{s_1}\right)^2.wt.\frac{1}{|a|\sqrt{\pi}} e^{-\left(\frac{D}{a}\right)^2}.\ln(wt) + D^4.\left(\frac{t_1}{s_1}\right)^3.wt.\frac{1}{a^2|a|\sqrt{\pi}} e^{-\left(\frac{D}{a}\right)^2} + c_1.wt\right] + c_3\,, \quad (38)$$

Appendix 2:

The mathematical proof of formula (7) is as follows, step by step.

From the general curvature formula, we would have:

$$\left|\frac{d^2P_c}{dwt^2}\right| = k.\left(1 + \left[\frac{dP_c}{dwt}\right]^2\right)^{3/2}, k > 0\,, \quad (1)$$

which results in:

$$\frac{d^2P_c}{dwt^2} = \pm k.\left(1 + \left[\frac{dP_c}{dwt}\right]^2\right)^{3/2}, \quad (2)$$

By the strategy of $\frac{d^2P_c}{dwt^2} > 0$, we would have:

$$\frac{d^2P_c}{dwt^2} = +k.\left(1 + \left[\frac{dP_c}{dwt}\right]^2\right)^{3/2}, \quad (3)$$

Then it can be reformed as:

$$\pm\left(\left(\frac{\frac{d^2P_c}{dwt^2}}{k}\right)^{2/3} - 1\right)^{0.5} = \frac{dP_c}{dwt}, \qquad (4)$$

By the strategy of $\frac{dP_c}{dwt} < 0$, we finally obtain:

$$\frac{dP_c}{dwt} + \left(\left(\frac{\frac{d^2P_c}{dwt^2}}{k}\right)^{2/3} - 1\right)^{0.5} = 0, \quad (5)$$

HOW TO CITE THIS ARTICLE
S. H. Ebrahimi, A. Safaei Ghadikolaei, S. H. Hashemi Petroudi, Proposing a Gaussian-Curvature-Based Nonlinear Natural Gas Pricing Model Incorporating Weather, Consumer Income, and Demand Dynamics, AUT J. Model. Simul., 58(1) (2025) 33-66.

DOI: 10.22060/miscj.2026.25114.5455