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Simulation of Singular Fourth- Order Partial Differential Equations Using the Fourier Transform Combined With Variational Iteration Method

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ABSTRACT

In this paper, we present a comparative study between the modified variational iteration method (MVIM) and a hybrid of Fourier transform and variational iteration method (FTVIM). The study outlines the efficiency and convergence of the two methods. The analysis is illustrated by investigating four singular partial differential equations with variable coefficients. The solution of singular partial differential equations usually needs a coordinate transformation in order to discard the singularity of the partial differential equation. Most often this transformation is not applicable and even does not exist. Therefore in this case the solution for the singular partial differential equation does not exist. In the present study the results of simulation for the singular partial differential equations with variable coefficients using the Fourier transform variational iteration method are compared with the results of simulation using the modified variational iteration method. The comparison shows that the effectiveness and accuracy of Fourier transform variational iteration method is more than that of the modified variational iteration method for the simulation of singular partial differential equations.

KEYWORDS

Fourier Transformation Modified Variational Iteration Method, Hybrid of Fourier Transform, Variational Iteration Method, Singular Partial Differential Equations With Variable Coefficients.

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1- INTRODUCTION

This paper outlines a reliable comparison between two powerful methods that were recently developed. The first is a hybrid of Fourier transform and variational iteration method (FTVIM) developed by S.S. Nourazar et al in [1]. The second is the modified variational iteration method (MVIM) developed by M. A. Noor et al in [2].

This paper is devoted to the study of the singular fourth - order parabolic partial differential equation with variable coefficient. It is well known in the literature that a wide class of problems arising in mathematics, physics and astrophysics and engineering sciences may be distinctively formulated as singular initial and boundary value problems. Singular fourth order parabolic partial differential equations govern the transverse vibrations of a homogeneous beam. Such types of equation arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory, see [4 - 10]. The studies of such problems have attracted the attention of many mathematicians and physicists. So finding a method with less amount of computational work in comparison with the previous methods may be useful. The study of fourth - order partial differential equations with variable coefficients are performed by solving the vibration equation of beam and shafts using the second - order finite difference method [4, 5]. However in their results the convergence of the solution with mesh refinement failed and therefore the accuracy of the results was limited to a certain amount of refined mesh. Wazwaz [6, 8, 9] studied the behavior of the fourth - order partial differential equations with variable coefficients. The application of such equations encounters in the study of deformation of beams and plates as well as the flow of viscoelastic and elastic fluids. Wazwaz [6, 8, 9] solved the governing equations using the semi - analytical method such as the Adomian decomposition method. However in the Adomian decomposition method the boundary conditions of the governing equation were totally ignored. This may cause an inaccurate results as well as incorporating enormous amounts of terms in series solution. In the present study we intend to use the FTVIM developed by Nourazar et al [1] to solve the singular fourth - order partial differential equations with variable coefficients that may arise in the study of the vibration of beams and plates as well as the flow of viscoelastic and elastic (Newtonian) fluid. These equations are solved by using the modified He's variational iteration method (MVIM) [2]. First, a coordinate transformation is used to resolve the singularity problem and then the MVIM [2] is applied to solve the governing equations. However, the advantage of the FTVIM is that there is no need to find a coordinate transformation to get rid of the singularity problem. We show the effectiveness of the FTVIM by solving four singular fourth - order parabolic differential equations with variable coefficients as case studies. The first case study problem is the vibration of an elastic beam with variable material properties along the axis of beam. In the second case study we solve the vibration of a thin two dimensional elastic plate with variable material properties along the two dimensions (x, y). In the third case study we solve the governing equation of a three dimensional vibrating plate with external sinusoidal forcing function. In the fourth case study we consider the vibration of an elastic beam with a sinusoidal variable property along the beam axis.

2- NUMERICAL APPLICATION

In this section, we apply the FTVIM proposed by S.S. Nourazar et al in [1] for solving the fourth order parabolic partial differential equations. For using FTVIM we construct a correction functional by using the Lagrange multipliers that calculated optimally via variational theory. And finally we can approximate the exact solutions. For the sake of comparison, we take the same examples as used in [2] and the numerical results are very encouraging and conclusive.

- **EXAMPLE 1**

Consider the following singular fourth parabolic partial differential

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4}=0 \qquad \frac{1}{2}<x<1 \quad , \quad t>0 \tag{1}$$

With initial conditions:

$$u(x,0)=0 \qquad \frac{1}{2}<x<1 \tag{2}$$

$$\frac{\partial}{\partial t}u(x,0)=1+\frac{x^5}{120} \qquad \frac{1}{2}<x<1 \tag{3}$$

First we take Fourier transform from equation (1) and immediately construct a correction functional as is done in reference [3]. Here û denotes the Fourier transform of u. Now for using FTVIM we construct a correction functional as[3]:

$$\hat{u}_{n+1}(\omega,t)=\hat{u}_n(\omega,t)+\int_0^t \lambda(\xi)\left\{\frac{\partial^2}{\partial\xi^2}\hat{u}_n(\omega,\xi)+F\left\{\frac{1}{x}\frac{\partial^4}{\partial x^4}u_n\right\}+\frac{1}{120}\frac{\partial^4}{\partial\omega^4}F\left\{\frac{\partial^4 u_n}{\partial x^4}\right\}\right\}d\xi \tag{4}$$

For detailed derivation of constructing the correction functional (Eq. (4)) one may refer to Wazwaz [3]. By using the variational principle and integrating by parts we may obtain the following. Integrating by parts and take first variation, we get $\lambda(\xi)=\xi-t$ as follows:

$$\Rightarrow \delta\hat{u}_{n+1}(\omega,t)=\delta\hat{u}_n(\omega,t)+\delta\int_0^t \lambda(\xi)\left\{\frac{\partial^2}{\partial\xi^2}\hat{u}_n(\omega,\xi)+F\left\{\frac{1}{x}\frac{\partial^4}{\partial x^4}u_n\right\}+\frac{1}{120}\frac{\partial^4}{\partial\omega^4}F\left\{\frac{\partial^4 u_n}{\partial x^4}\right\}\right\}d\xi \tag{5}$$

$$\Rightarrow \delta\hat{u}_{n+1}(\omega,t)=\delta\hat{u}_n(\omega,t)+\delta\int_0^t \lambda(\xi)\left\{\frac{\partial^2}{\partial\xi^2}\hat{u}_n(\omega,\xi)\right\}d\xi \tag{6}$$

$$\Rightarrow \delta\hat{u}_{n+1}=\delta\hat{u}_n+\delta\left\{\lambda(\xi)\frac{\partial}{\partial\xi}\hat{u}_n-\lambda'(\xi)\hat{u}_n+\int\lambda''(\xi)\hat{u}_n d\xi\right\} \tag{7}$$

$$\Rightarrow \delta\hat{u}_{n+1}=\left[1-\lambda'(\xi)\right]\delta\hat{u}_n+\delta\lambda(\xi)\frac{\partial}{\partial\xi}\hat{u}_n+\int_0^t\lambda''(\xi)\delta\hat{u}_n d\xi \tag{8}$$

$$\Rightarrow\begin{cases}1-\lambda'(\xi)\big|_{\xi=t}=0\\ \lambda(\xi)\big|_{\xi=t}=0\\ \lambda''(\xi)\big|_{\xi=t}=0\end{cases}\qquad\Rightarrow\lambda(\xi)=\xi-t \tag{9}$$

Assuming $u_0(x, t)$, using the same method as used in [3], and substituting for the value of $\lambda(\xi)=\xi-t$, into Eq.(4) the successive approximation $\hat{u}_{n+1}(\omega,t)$ are obtained as follows:

$$\hat{u}_1(\omega,t)=\hat{u}_0(\omega,t)+\int_0^t(\xi-t)\left\{\frac{\partial^2}{\partial\xi^2}\hat{u}_0(\omega,\xi)+F\left\{\frac{1}{x}\frac{\partial^4}{\partial x^4}u_0\right\}+\frac{1}{120}\frac{\partial^4}{\partial\omega^4}F\left\{\frac{\partial^4 u_0}{\partial x^4}\right\}\right\}d\xi \tag{10}$$

$$\hat{u}_2(\omega,t)=\hat{u}_1(\omega,t)+\int_0^t(\xi-t)\left\{\frac{\partial^2}{\partial\xi^2}\hat{u}_1(\omega,\xi)+F\left\{\frac{1}{x}\frac{\partial^4}{\partial x^4}u_1\right\}+\frac{1}{120}\frac{\partial^4}{\partial\omega^4}F\left\{\frac{\partial^4 u_1}{\partial x^4}\right\}\right\}d\xi \tag{11}$$

$$\hat{u}_3(\omega,t)=\hat{u}_2(\omega,t)+\int_0^t(\xi-t)\left\{\frac{\partial^2}{\partial\xi^2}\hat{u}_2(\omega,\xi)+F\left\{\frac{1}{x}\frac{\partial^4}{\partial x^4}u_2\right\}+\frac{1}{120}\frac{\partial^4}{\partial\omega^4}F\left\{\frac{\partial^4 u_2}{\partial x^4}\right\}\right\}d\xi \tag{12}$$

$$\Rightarrow\hat{u}_{n+1}(\omega,t)=\hat{u}_n(\omega,t)+\int_0^t(\xi-t)\left\{\frac{\partial^2}{\partial\xi^2}\hat{u}_n(\omega,\xi)+F\left\{\frac{1}{x}\frac{\partial^4}{\partial x^4}u_n\right\}+\frac{1}{120}\frac{\partial^4}{\partial\omega^4}F\left\{\frac{\partial^4 u_n}{\partial x^4}\right\}\right\}d\xi \tag{13}$$

For obtaining $u_1(x,t)$ first we calculate followings:

$$u_0(x,t)=\left(1+\frac{x^5}{120}\right)t \tag{14}$$

Eq. (14) is obtained by integrating the Neumann initial condition (Eq. (3)) and considering as $u_0(x,t)$.

$$\hat{u}_0(\omega,t)=\frac{1}{120}\frac{\left(I\pi Dirac(5,\omega)\omega^6-120+120\pi Dirac(\omega)\omega^6-120I\omega^5\right)}{\omega^6}t \tag{15}$$

$$\frac{1}{x}\frac{\partial^4}{\partial x^4}\left(1+\frac{x^5}{120}\right)t=t \tag{16}$$

$$\frac{\partial^4}{\partial x^4}\left(1+\frac{x^5}{120}\right)t = xt \tag{17}$$

$$F\left\{\frac{1}{x}\frac{\partial^4}{\partial x^4}u_0\right\} = \left(\pi Dirac(\omega)-\frac{I}{\omega}\right)t \tag{18}$$

$$\frac{\partial^4}{\partial\omega^4}F\left\{\frac{\partial^4 u_0}{\partial x^4}\right\} = \left(\frac{I\left(\pi Dirac(5,\omega)\omega^2+8\pi Dirac(4,\omega)\omega+12\pi Dirac(3,\omega)\right)}{\omega^2} - \frac{8I\left(\pi Dirac(4,\omega)\omega^2+6\pi Dirac(3,\omega)\omega+6\pi Dirac(2,\omega)\right)}{\omega^3} + \frac{36I\left(\pi Dirac(3,\omega)\omega^2+4\pi Dirac(2,\omega)\omega+2\pi Dirac(1,\omega)\right)}{\omega^4} - \frac{96I\left(\pi Dirac(2,\omega)\omega^2+2\pi Dirac(1,\omega)\omega\right)}{\omega^5} + \frac{120I\left(\pi Dirac(1,\omega)\omega^2+I\right)}{\omega^6}\right)t \tag{19}$$

$$\hat{u}_1(\omega,t) = \frac{1}{120}\frac{\begin{pmatrix} I\pi Dirac(5,\omega)\omega^6-120+ \\ 120\pi Dirac(\omega)\omega^6-120I\omega^5\end{pmatrix}}{\omega^6}t + \int_0^t(\xi-t)\left(\left(\pi Dirac(\omega)-\frac{I}{\omega}\right)\xi + \frac{1}{120}\left(\frac{I\begin{pmatrix}\pi Dirac(5,\omega)\omega^2+8\pi Dirac(4,\omega)\omega \\ +12\pi Dirac(3,\omega)\end{pmatrix}}{\omega^2} \frac{-8I\left(\pi Dirac(4,\omega)\omega^2+6\pi Dirac(3,\omega)\omega+6\pi Dirac(2,\omega)\right)}{\omega^3} + \frac{36I\left(\pi Dirac(3,\omega)\omega^2+4\pi Dirac(2,\omega)\omega+2\pi Dirac(1,\omega)\right)}{\omega^4} - \frac{96I\left(\pi Dirac(2,\omega)\omega^2+2\pi Dirac(1,\omega)\omega\right)}{\omega^5} + \frac{120I\left(\pi Dirac(1,\omega)\omega^2+I\right)}{\omega 6}\right)\xi\right)d\xi \rightarrow \hat{u}_1(\omega,t) = \frac{1}{120}\frac{\left(I\pi Dirac(5,\omega)\omega^6-120+120\pi Dirac(\omega)\omega^6-120I\omega^5\right)}{\omega^6}t - \frac{1}{6}\left(\pi Dirac(\omega)-\frac{I}{\omega}\right) + \frac{1}{120}\left(\frac{I\left(\pi Dirac(5,\omega)\omega^2+8\pi Dirac(4,\omega)\omega+12\pi Dirac(3,\omega)\right)}{\omega^2} - \frac{8I\left(\pi Dirac(4,\omega)\omega^2+6\pi Dirac(3,\omega)\omega+6\pi Dirac(2,\omega)\right)}{\omega^3} + \frac{36I\left(\pi Dirac(3,\omega)\omega^2+4\pi Dirac(2,\omega)\omega+2\pi Dirac(1,\omega)\right)}{\omega^4} - \frac{96I\left(\pi Dirac(2,\omega)\omega^2+2\pi Dirac(1,\omega)\omega\right)}{\omega^5} + \frac{120I\left(\pi Dirac(1,\omega)\omega^2+I\right)}{\omega^6}\right)\Bigg)t^3 \tag{20}$$

$$\rightarrow \hat{u}_1(\omega,t) = \frac{1}{120}\frac{\left(I\pi Dirac(5,\omega)\omega^6-120+120\pi Dirac(\omega)\omega^6-120I\omega^5\right)}{\omega^6}\times\left(t-\frac{t^3}{6}\right) \tag{21}$$

Using the Maple package the inverse Fourier transform, $u_1(x,t)$ is:

$$u_1(x,t) = \left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{6}\right) \tag{22}$$

Here, we are taking the inverse Fourier transform from equation (21) using the Maple package. After some simple manipulations we get Eq. (22)

For calculating $u_2(x,t)$, we use the correction functional of Eq. (4) using the value of $\hat{u}_1(\omega,t)$ and taking the inverse Fourier transform as:

$$\frac{\partial^2}{\partial\xi^2}\hat{u}_1(\omega,\xi) = \frac{1}{120}\frac{(I\pi\, Dirac(5,\omega)\,\omega^6 - 120 + 120\,\pi\, Dirac(\omega)\,\omega^6 - 120\, I\,\omega^5}{\omega^6}\times(-\xi) \tag{23}$$

$$F\left\{\frac{1}{x}\frac{\partial^4}{\partial x^4}u_1\right\} = \left(\pi Dirac(\omega)-\frac{I}{\omega}\right)\left(t-\frac{t^3}{6}\right) \tag{24}$$

$$\frac{\partial^4}{\partial\omega^4}F\left\{\frac{\partial^4}{\partial x^4}u_1\right\}=\left(\frac{I(\pi Dirac(5,\omega)\,\omega^2+8\,\pi Dirac(4,\omega)\,\omega-12\pi Dirac(3,\omega))}{\omega^2}-\frac{8I\,(\pi\,Dirac\,(4,\omega)\,\omega^2+6\pi\,Dirac\,(3,\omega)\,\omega+6\pi\,Dirac\,(2,\omega))}{\omega^3}\right.$$
$$\left.+\frac{36I\,(\pi\,Dirac\,(3,\omega)\,\omega^2+4\pi\,Dirac\,(2,\omega)\,\omega+2\pi\,Dirac\,(1,\omega))}{\omega^4}+\frac{96I(\pi Dirac(2,\omega)\omega^2+2\pi\,Dirac\,(1,\omega)\omega)}{\omega^5}+\frac{120I(\pi Dirac(1,\omega)\omega^2+I)}{\omega^6}\right)$$
$$\times\left(t-\frac{t^3}{6}\right) \tag{25}$$

Using correction functional and the Maple package, the inverse Fourier transform, $u_2(x,t)$ is:

$$u_2(x,t)=\left(1+\frac{x^5}{120}\right)\times\left(t-\frac{t^3}{6}+\frac{t^5}{120}\right) \tag{26}$$

Here, we are using the correction functional of Eq. (4) and taking the inverse Fourier transform one gets Eq. (26).

For $u_3(x,t)$:

$$\frac{\partial^2}{\partial\xi^2}\hat{u}_2(\omega,t)=\frac{1}{120}\frac{\left(I\,\pi Dirac\,(5,\omega)\omega^6-120+120\pi Dirac\,(\omega)\omega^6-120I\,\omega^5\right)}{\omega^6}\times\left(-\xi+\frac{1}{6}\xi^3\right) \tag{27}$$

$$F\left\{\frac{1}{x}\frac{\partial^4}{\partial x^4}u_2\right\}=\left(\pi Dirac\,(\omega)-\frac{I}{\omega}\right)\times\left(t-\frac{t^3}{6}+\frac{t^5}{120}\right) \tag{28}$$

$$\frac{\partial^4}{\partial\omega^4}F\left\{\frac{\partial^4}{\partial x^4}u_2\right\}=\left(\frac{I\left(\pi Dirac(5,\omega)\omega^2+8\pi Dirac(4,\omega)\omega+12\pi Dirac(3,\omega)\right)}{\omega^2}-\frac{8I\left(\pi Dirac(4,\omega)\omega^2+6\pi Dirac(3,\omega)\omega+6\pi Dirac(2,\omega)\right)}{\omega^3}+\frac{36I\left(\pi Dirac(3,\omega)\omega^2\right.}{}\right.$$
$$\left.\frac{\left.+4\pi Dirac(2,\omega)\omega+2\pi Dirac(1,\omega)\right)}{\omega^4}-\frac{96I\left(\pi Dirac(2,\omega)\omega^2+2\pi Dirac(1,\omega)\omega\right)}{\omega^5}+\frac{120I\left(\pi Dirac(1,\omega)\omega^2+I\right)}{\omega^6}\right)\times\left(t-\frac{t^3}{6}+\frac{t^5}{120}\right) \tag{29}$$

Using correction functional and the Maple package the Fourier transform and inverse Fourier transform, $u_3(x,t)$ is:

$$u_3(x,t)=\left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{6}+\frac{t^5}{120}-\frac{t^7}{5040}\right) \tag{30}$$

And so on. The Taylor series expansion for sin (t) is written as:

$$\sin(t)=\sum_{i=0}^{n}\frac{(-1)^i\,t^{(2i+1)}}{(2i+1)!}=t-\frac{t^3}{6}+\frac{t^5}{120}-\frac{t^7}{5040}+\cdots \tag{31}$$

And:

$$u(x,t)=\lim_{n\to\infty}u_n \tag{32}$$

By substituting Eq (31) into Eq (30) thus Eq (30) can ultimately be reduced to:

$$u(x,t)=\lim_{n\to\infty}u_n=\left(1+\frac{x^5}{120}\right)\sin(t) \tag{33}$$

Which, it is the exact solution.

- **EXAMPLE 2**

Consider the following singular fourth parabolic partial differential equation in two space variables:

$$\frac{\partial^2u}{\partial t^2}+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4u}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4u}{\partial y^4}=0 \tag{34}$$

With initial conditions:

$$u(x,y,0)=0, \qquad \frac{\partial u}{\partial t}(x,y,0)=2+\frac{x^6}{6!}+\frac{y^6}{6!} \tag{35}$$

Now using FTVIM we construct a correction functional as:

$$\hat{u}_{n+1}(\omega,y,t)=\hat{u}_n(\omega,y,t)+\int_0^t \lambda(\xi)\left\{\frac{\partial^2\hat{u}_n(\omega,y,\xi)}{\partial\xi^2}+F\left\{2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u_n(x,y,\xi)}{\partial x^4}\right\}+F\left\{2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u_n(x,y,\xi)}{\partial y^4}\right\}\right\}d\xi \tag{36}$$

Integrating by parts and taking the first variation, we get $\lambda(\xi)=\xi-t$ as follows:

$$\Rightarrow \delta\hat{u}_{n+1}(\omega,y,t)=\delta\hat{u}_n(\omega,y,t)+\delta\int_0^t \lambda(\xi)\left\{\frac{\partial^2\hat{u}_n(\omega,y,\xi)}{\partial\xi^2}+F\left\{2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u_n(x,y,\xi)}{\partial x^4}\right\}+F\left\{2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u_n(x,y,\xi)}{\partial y^4}\right\}\right\}d\xi \tag{37}$$

$$\Rightarrow \delta\hat{u}_{n+1}(\omega,y,t)=\delta\hat{u}_n(\omega,y,t)+\delta\int_0^t \lambda(\xi)\left\{\frac{\partial^2\hat{u}_n(\omega,y,\xi)}{\partial\xi^2}\right\}d\xi \tag{38}$$

$$\Rightarrow \delta\hat{u}_{n+1}=\delta\hat{u}_n+\delta\left\{\lambda(\xi)\frac{\partial\hat{u}_n}{\partial\xi}-\lambda'(\xi)\hat{u}_n+\int\lambda''(\xi)\hat{u}_n\,d\xi\right\} \tag{39}$$

$$\Rightarrow \delta\hat{u}_{n+1}=\left[1-\lambda'(\xi)\right]\delta\hat{u}_n+\delta\lambda(\xi)\frac{\partial\hat{u}_n}{\partial\xi}+\int_0^t\lambda''(\xi)\hat{u}_n\,d\xi \tag{40}$$

$$\Rightarrow 1-\lambda'(\xi)\big|_{\xi=t}=0 \quad \lambda(\xi)\big|_{\xi=t}=0 \quad \lambda''(\xi)\big|_{\xi=t}=0 \quad \Rightarrow \lambda(\xi)=\xi-t \tag{41}$$

Assuming $u_0(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t$, using the same method as used in [3], and substituting for the value of $\lambda(\xi)=\xi-t$ into Eq. (36) the successive approximation $\hat{u}_{n+1}(\omega,y,t)$ are obtained as follows:

$$\hat{u}_1(\omega,y,t)=\hat{u}_0(\omega,y,t)+\int_0^t \lambda(\xi)\left\{\frac{\partial^2\hat{u}_0(\omega,y,\xi)}{\partial\xi^2}+F\left\{2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u_0(x,y,\xi)}{\partial x^4}\right\}+F\left\{2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u_0(x,y,\xi)}{\partial y^4}\right\}\right\}d\xi \tag{42}$$

$$\hat{u}_2(\omega,y,t)=\hat{u}_1(\omega,y,t)+\int_0^t \lambda(\xi)\left\{\frac{\partial^2\hat{u}_1(\omega,y,\xi)}{\partial\xi^2}+F\left\{2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u_1(x,y,\xi)}{\partial x^4}\right\}+F\left\{2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u_1(x,y,\xi)}{\partial y^4}\right\}\right\}d\xi \tag{43}$$

$$\hat{u}_3(\omega,y,t)=\hat{u}_2(\omega,y,t)+\int_0^t \lambda(\xi)\left\{\frac{\partial^2\hat{u}_2(\omega,y,\xi)}{\partial\xi^2}+F\left\{2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u_2(x,y,\xi)}{\partial x^4}\right\}+F\left\{2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u_2(x,y,\xi)}{\partial y^4}\right\}\right\}d\xi \tag{44}$$

$$\hat{u}_{n+1}(\omega,y,t)=\hat{u}_n(\omega,y,t)+\int_0^t \lambda(\xi)\left\{\frac{\partial^2\hat{u}_n(\omega,y,\xi)}{\partial\xi^2}+F\left\{2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u_n(x,y,\xi)}{\partial x^4}\right\}+F\left\{2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u_n(x,y,\xi)}{\partial y^4}\right\}\right\}d\xi \tag{45}$$

For obtaining $u_1(x,y,t)$, first we calculate followings:

$$\frac{\partial^4}{\partial x^4}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t=\frac{1}{2}x^2t \tag{46}$$

$$\frac{\partial^4}{\partial y^4}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t=\frac{1}{2}y^2t \tag{47}$$

$$\frac{1}{2}x^2t\times 2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)=x^2t\left(\frac{1}{x^2}+\frac{x^4}{720}\right)=t+\frac{1}{720}tx^6 \tag{48}$$

$$\frac{1}{2}y^2 t\times 2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)=y^2 t\left(\frac{1}{y^2}+\frac{y^4}{720}\right)=t+\frac{1}{720}ty^6 \tag{49}$$

$$F\left\{2+\frac{x^6}{6!}+\frac{y^6}{6!}\right\}t=\frac{1}{720}\left(\frac{1440\pi Dirac(\omega)\omega^7+\pi Dirac(\omega)\omega^7 y^6-\pi Dirac(6,\omega)\omega^7+720I-1440I\omega^6-I\omega^6 y^6}{\omega^7}\right)t \tag{50}$$

$$F\left\{t+\frac{1}{720}tx^6\right\}=\frac{1}{720}\left(\frac{-\pi Dirac(6,\omega)\omega^7+720\pi Dirac(\omega)\omega^7+720I-720I\omega^6}{\omega^7}\right)t \tag{51}$$

$$F\left\{t+\frac{1}{720}ty^6\right\}=\frac{1}{360}\pi Dirac(\omega)\left(720+y^6\right)t \tag{52}$$

$$\hat{u}_1(\omega,y,t)=\frac{1}{720}\left(\frac{1440\pi Dirac(\omega)\omega^7+\pi Dirac(\omega)\omega^7 y^6-\pi Dirac(6,\omega)\omega^7+720I-1440I\omega^6-I\omega^6 y^6}{\omega^7}\right)t+\int_0^t(\xi-t)\left(\frac{\xi}{720}\right.$$
$$\left.\frac{-\pi Dirac(6,\omega)\omega^7+720\pi Dirac(\omega)\omega^7+720I-720I\omega^6}{\omega^7}+\frac{\xi}{360}\left(720+y^6\right)\pi Dirac(\omega)\right)d\xi \tag{53}$$

Using the Maple package, the inverse Fourier transform, $u_1(x,y,t)$ is:

$$u_1(x,y,t)=-\frac{1}{4320}t\left(-6+t^2\right)\left(1440+x^6+y^6\right)=\left(t-\frac{t^3}{6}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right) \tag{54}$$

For $u_2(x,y,t)$

$$\frac{\partial^4}{\partial x^4}\left(u_1(x,y,t)\right)=\frac{\partial^4}{\partial x^4}\left(\left(t-\frac{t^3}{6}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\right)=\frac{1}{2}\left(t-\frac{t^3}{6}\right)x^2 \tag{55}$$

$$\frac{\partial^4}{\partial y^4}\left(u_1(x,y,t)\right)=\frac{\partial^4}{\partial y^4}\left(\left(t-\frac{t^3}{6}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\right)=\frac{1}{2}\left(t-\frac{t^3}{6}\right)y^2 \tag{56}$$

$$\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\left(\frac{\partial^4}{\partial x^4}\left(u_1(x,y,t)\right)\right)=\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\left(\frac{1}{2}\left(t-\frac{t^3}{6}\right)x^2\right)=\frac{t}{2}-\frac{t^3}{12}+\frac{tx^6}{1440}-\frac{t^3x^6}{8640} \tag{57}$$

$$\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\left(\frac{\partial^4}{\partial y^4}\left(u_1(x,y,t)\right)\right)=\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\left(\frac{1}{2}\left(t-\frac{t^3}{6}\right)y^2\right)=\frac{t}{2}-\frac{t^3}{12}+\frac{ty^6}{1440}-\frac{t^3y^6}{8640} \tag{58}$$

Using correction functional and the Maple package, the Fourier transform and inverse Fourier transform, $u_2(x, y, t)$ is:

$$u_2(x,y,t)=\frac{1}{86400}t\left(120+t^4-20t^2\right)\left(1440+x^6+y^6\right)=\left(t-\frac{t^3}{6}+\frac{t^5}{120}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right) \tag{59}$$

For $u_3(x,y,t)$

$$\frac{\partial^4}{\partial x^4}\left(u_2(x,y,t)\right)=\frac{\partial^4}{\partial x^4}\left(\left(t-\frac{t^3}{6}+\frac{t^5}{120}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\right)=\frac{1}{2}\left(t-\frac{t^3}{6}+\frac{t^5}{120}\right)x^2 \tag{60}$$

$$\frac{\partial^4}{\partial y^4}\left(u_2(x,y,t)\right)=\frac{\partial^4}{\partial y^4}\left(\left(t-\frac{t^3}{6}+\frac{t^5}{120}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\right)=\frac{1}{2}\left(t-\frac{t^3}{6}+\frac{t^5}{120}\right)y^2 \tag{61}$$

$$\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\left(\frac{\partial^4}{\partial x^4}\left(u_2(x,y,t)\right)\right)=\frac{t}{2}-\frac{t^3}{12}+\frac{t^5}{240}+\frac{tx^6}{1440}-\frac{t^3x^6}{8640}+\frac{t^5x^6}{172800} \tag{62}$$

$$\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\left(\frac{\partial^4}{\partial y^4}\left(u_2(x,y,t)\right)\right)=\frac{t}{2}-\frac{t^3}{12}+\frac{t^5}{240}+\frac{ty^6}{1440}-\frac{t^3y^6}{8640}+\frac{t^5y^6}{172800} \tag{63}$$

Using correction functional and the Maple package, the Fourier transform and inverse Fourier transform, $u_3(x, y, t)$ is:

$$u_3(x,y,t)=-\frac{1}{3628800}t\left(-5040-42t^4+840t^2+t^6\right)\left(1440+x^6+y^6\right)=\left(t-\frac{t^3}{6}+\frac{t^5}{120}-\frac{t^7}{5040}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right) \tag{64}$$

And so on. The Taylor series expansion for sin(t)is written as:

$$\sin(t)=\sum_{i=0}^{n}\frac{(-1)^i\,t^{(2i+1)}}{(2i+1)!}=t-\frac{t^3}{6}+\frac{t^5}{120}-\frac{t^7}{5040}+\ldots \tag{65}$$

And

$$u(x,y,t)=\lim_{n\to\infty}u_n \tag{66}$$

By substituting Eq. (64) and Eq. (65) into Eq. (66) thus Eq. (64) can ultimately be reduced to:

$$u(x,y,t)=\lim_{n\to\infty}u_n=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\times\sin(t) \tag{67}$$

Which, it is the exact solution of Eq. (34).

- **EXAMPLE 3**

Considering the following three dimensional non - homogenous singular partial differential equation, we solve this equation with FTVIM.

$$\frac{\partial^2 u}{\partial t^2}-\left(\frac{1}{4!z}\times\frac{\partial^4 u}{\partial x^4}+\frac{1}{4!x}\times\frac{\partial^4 u}{\partial y^4}+\frac{1}{4!y}\times\frac{\partial^4 u}{\partial z^4}\right)=-\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right]\cos(t) \tag{68}$$

With initial condition:

$$u(x,y,z,0)=\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \tag{69}$$

Now we construct a correction functional as:

$$\hat{u}_{n+1}(\omega,y,z,t)=\hat{u}_{n+1}(\omega,y,z,t)+\int_0^t\lambda(\xi)\times F\left\{\begin{array}{l}-\left(\begin{array}{l}\frac{1}{4!z}\times\frac{\partial^4 u_n(\omega,y,z,t)}{\partial x^4}+\frac{1}{4!x}\times\frac{\partial^4 u_n(\omega,y,z,t)}{\partial y^4}\\+\frac{1}{4!y}\times\frac{\partial^4 u_n(\omega,y,z,t)}{\partial z^4}\end{array}\right)\\+\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right]\cos(\xi)+\frac{\partial^2 u_n(\omega,y,z,t)}{\partial\xi^2}\end{array}\right\}d\xi \tag{70}$$

Integrating by parts and taking the first variation, we get $\lambda(\xi)=\xi-t$ as follows:

$$\Rightarrow\delta\hat{u}_{n+1}=\delta\hat{u}_n+\delta\int_0^t\lambda(\xi)\times F\left\{\begin{array}{l}-\left(\frac{1}{4!z}\times\frac{\partial^4 u_n}{\partial x^4}+\frac{1}{4!x}\times\frac{\partial^4 u_n}{\partial y^4}+\frac{1}{4!y}\times\frac{\partial^4 u_n}{\partial z^4}\right)\\+\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right]\cos(\xi)+\frac{\partial^2 u_n}{\partial\xi^2}\end{array}\right\}d\xi \tag{71}$$

$$\Rightarrow\delta\hat{u}_{n+1}=\delta\hat{u}_n+\delta\int_0^t\lambda(\xi)\times F\left\{\frac{\partial^2 u_n}{\partial\xi^2}\right\}d\xi+\delta\int_0^t\lambda(\xi)\times F\left\{\begin{array}{l}-\left(\frac{1}{4!z}\times\frac{\partial^4 u_n}{\partial x^4}+\frac{1}{4!x}\times\frac{\partial^4 u_n}{\partial y^4}+\frac{1}{4!y}\times\frac{\partial^4 u_n}{\partial z^4}\right)\\+\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right]\cos(\xi)\end{array}\right\}d\xi \tag{72}$$

$$\Rightarrow \delta \hat{u}_{n+1} = \delta \hat{u}_n + \delta \int_0^t \lambda(\xi) \times \frac{\partial^2 \hat{u}_n}{\partial \xi^2} d\xi \tag{73}$$

$$\Rightarrow \delta \hat{u}_{n+1} = \delta \hat{u}_n + \delta \left[\lambda(\xi) \times \frac{\partial \hat{u}_n|}{\partial \xi} - \lambda'(\xi)\hat{u}_n + \int_0^t \lambda''(\xi)\hat{u}_n d\xi \right] \tag{74}$$

$$\Rightarrow \delta \hat{u}_{n+1} = [1 - \lambda'(\xi)]\delta \hat{u}_n + \delta \left(\lambda(\xi) \times \frac{\partial \hat{u}_n}{\partial \xi} \right) + \delta \int_0^t \lambda''(\xi)\hat{u}_n d\xi \tag{75}$$

$$\Rightarrow \begin{cases} 1 - \lambda'(\xi)\big|_{\xi=t} = 0 \\ \lambda(\xi)\big|_{\xi=t} = 0 \qquad \Rightarrow \lambda(\xi) = \xi - t \\ \lambda''(\xi)\big|_{\xi=t} = 0 \end{cases} \tag{76}$$

Assuming u_0 (x, y, z, t) = u (x, y, z, 0), using the same method as used in [3], and substituting for the value of $\lambda(\xi) = \xi - t$, and using Eq. (70) the successive approximation $\hat{u}_{n+1}$ (ϖ, y, z, t) are obtained as follows:

$$\hat{u}_1 = \hat{u}_0 + \int_0^t (\xi - t) \times F\left\{ \frac{\partial^2 u_0}{\partial \xi^2} - \left(\frac{1}{4!z} \times \frac{\partial^4 u_0}{\partial x^4} + \frac{1}{4!x} \times \frac{\partial^4 u_0}{\partial y^4} + \frac{1}{4!y} \times \frac{\partial^4 u_0}{\partial z^4} \right) + \left[\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right] \cos(\xi) \right\} d\xi \tag{77}$$

$$\hat{u}_2 = \hat{u}_1 + \int_0^t (\xi - t) \times F\left\{ \frac{\partial^2 u_1}{\partial \xi^2} - \left(\frac{1}{4!z} \times \frac{\partial^4 u_1}{\partial x^4} + \frac{1}{4!x} \times \frac{\partial^4 u_1}{\partial y^4} + \frac{1}{4!y} \times \frac{\partial^4 u_1}{\partial z^4} \right) + \left[\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right] \cos(\xi) \right\} d\xi \tag{78}$$

$$\hat{u}_3 = \hat{u}_2 + \int_0^t (\xi - t) \times F\left\{ \frac{\partial^2 u_2}{\partial \xi^2} - \left(\frac{1}{4!z} \times \frac{\partial^4 u_2}{\partial x^4} + \frac{1}{4!x} \times \frac{\partial^4 u_2}{\partial y^4} + \frac{1}{4!y} \times \frac{\partial^4 u_2}{\partial z^4} \right) + \left[\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right] \cos(\xi) \right\} d\xi \tag{79}$$

$$\hat{u}_{n+1} = \hat{u}_n + \int_0^t (\xi - t) \times F\left\{ \frac{\partial^2 u_n}{\partial \xi^2} - \left(\frac{1}{4!z} \times \frac{\partial^4 u_n}{\partial x^4} + \frac{1}{4!x} \times \frac{\partial^4 u_n}{\partial y^4} + \frac{1}{4!y} \times \frac{\partial^4 u_n}{\partial z^4} \right) + \left[\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right] \cos(\xi) \right\} d\xi \tag{80}$$

For obtaining $\hat{u}_1$ (ϖ, y, z, t), first we calculate following terms:

$$\frac{\partial^2}{\partial t^2} u_0(x, y, z, t) = 0 \qquad \frac{\partial^4}{\partial x^4} u_0(x, y, z, t) = \frac{24z}{x^5} \qquad \frac{\partial^4}{\partial y^4} u_0(x, y, z, t) = \frac{24x}{y^5}$$

$$\frac{\partial^4}{\partial z^4} u_0(x, y, z, t) = \frac{24y}{z^5} \qquad \frac{1}{4!z} \times \frac{\partial^4}{\partial x^4} u_0(\omega, y, z, t) = \frac{1}{x^5} \tag{81}$$

$$\frac{1}{4!x} \times \frac{\partial^4}{\partial y^4} u_0(\omega, y, z, t) = \frac{1}{y^5}$$

$$\frac{1}{4!y} \times \frac{\partial^4}{\partial z^4} u_0(\omega, y, z, t) = \frac{1}{z^5}$$

$$F\left\{ \frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right\} = F\left\{ \frac{1}{x^5} \right\} + \frac{(z+y)(-z^4 + z^3 y - z^2 y^2 + z y^3 - y^4)(-\pi\omega Dirac(\omega) + I)}{y^5 z^5 \omega} \tag{82}$$

Using the Maple package, the inverse Fourier Transform, $u_1(x, y, z, t)$ is:

$$u_1(x, y, z, t) = \frac{1}{2}\frac{t^2}{x^5} - \frac{1}{x^5} + \frac{\cos(t)}{x^5} - \frac{1}{z^5} + \frac{x\cos(t)}{y} + \frac{1}{2}\frac{t^2}{y^5} + \frac{1}{2}\frac{t^2}{z^5} + \frac{\cos(t)}{y^5} + \frac{\cos(t)}{z^5} + \frac{y\cos(t)}{z} - \frac{1}{y^5} + \frac{z\cos(t)}{x} \rightarrow u_1(x, y, z, t) = \left(\frac{x}{y} + \right.$$

$$\left. \frac{y}{z} + \frac{z}{x} + \frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right)\cos(t) - \left(\frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right) + \frac{t^2}{2}\left(\frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right) \rightarrow u_1(x, y, z, t) = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5} \right)\cos(t) \tag{83}$$

For $u_2(x, y, z, t)$

$$u_1(x,y,z,t)=\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)\cos(t)-\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)\left(1-\frac{t^2}{2}\right)\ \frac{\partial^2}{\partial t^2}u_1(x,y,z,t)=\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}-\cos(t)$$
$$\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)-\cos(t)\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right) \tag{84}$$

$$\frac{\partial^4}{\partial x^4}u_1(x,y,z,t)=\frac{24z\cos(t)}{x^5}+\frac{840t^2}{x^9}+\frac{1680\cos(t)}{x^9}-\frac{1680}{x^9}$$
$$\frac{\partial^4}{\partial y^4}u_1(x,y,z,t)=\frac{24x\cos(t)}{y^5}+\frac{840t^2}{y^9}+\frac{1680\cos(t)}{y^9}-\frac{1680}{y^9} \tag{85}$$

$$\frac{\partial^4}{\partial z^4}u_1(x,y,z,t)=\frac{24y\cos(t)}{z^5}+\frac{840t^2}{z^9}+\frac{1680\cos(t)}{z^9}-\frac{1680}{z^9}$$
$$\frac{1}{4!z}\frac{\partial^4}{\partial x^4}u_1(x,y,z,t)=\frac{\cos(t)}{x^5}+\frac{35t^2}{zx^9}+\frac{70\cos(t)}{zx^9}-\frac{70}{zx^9}$$
$$\frac{1}{4!x}\frac{\partial^4}{\partial y^4}u_1(x,y,z,t)=\frac{\cos(t)}{y^5}+\frac{35t^2}{xy^9}+\frac{70\cos(t)}{xy^9}-\frac{70}{xy^9}$$
$$\frac{1}{4!y}\frac{\partial^4}{\partial z^4}u_1(x,y,z,t)=\frac{\cos(t)}{z^5}+\frac{35t^2}{yz^9}+\frac{70\cos(t)}{yz^9}-\frac{70}{yz^9} \tag{86}$$

Using the Maple package, the inverse Fourier Transform, $u_2(x,y,z,t)$ is:

$$u_2(x,y,z,t)=\frac{1}{12}\frac{1}{x^9y^9z^9}\begin{pmatrix}-420y^9z^8t^2-420x^8z^9t^2-420x^9y^8t^2-840x^8z^9\cos(t)\\ -840x^9y^8\cos(t)-840y^9z^8\cos(t)+35x^9y^8t^4+35x^8z^9t^4\\ +35y^9z^8t^4+840x^9y^8+840x^8z^9+840y^9z^8\\ +12x^{10}z^9y^8\cos(t)+12x^9z^8y^{10}\cos(t)+12x^8y^9z^{10}\cos(t)\end{pmatrix}$$
$$\to u_2(x,y,z,t)=\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{70}{zx^9}-\frac{70}{yz^9}-\frac{70}{xy^9}\right)\cos(t)+70\left(\frac{1}{zx^9}+\frac{1}{yz^9}+\frac{1}{xy^9}\right)-\frac{70}{2!}t^2\left(\frac{1}{zx^9}+\frac{1}{yz^9}+\frac{1}{xy^9}\right)+\frac{70}{4!}t^4\left(\frac{1}{zx^9}+\frac{1}{yz^9}+\frac{1}{xy^9}\right)$$
$$\to u_2(x,y,z,t)=\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{70}{zx^9}-\frac{70}{yz^9}-\frac{70}{xy^9}\right)\cos(t)+70\left(\frac{1}{zx^9}+\frac{1}{yz^9}+\frac{1}{xy^9}\right)\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4\right) \tag{87}$$

For $u_3(x,y,z,t)$

$$u_2(x,y,z,t)=\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{70}{zx^9}-\frac{70}{yz^9}-\frac{70}{xy^9}\right)\cos(t)+70\left(\frac{1}{zx^9}+\frac{1}{yz^9}+\frac{1}{xy^9}\right)\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4\right)\ \frac{\partial^2 u_2(x,y,z,t)}{\partial t^2}=70$$
$$\left(\frac{1}{zx^9}+\frac{1}{yz^9}+\frac{1}{xy^9}\right)\left(-1+\frac{t^2}{2!}\right)-\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{70}{zx^9}-\frac{70}{yz^9}-\frac{70}{xy^9}\right)\cos(t) \tag{88}$$

$$\frac{\partial^4}{\partial x^4}u_2(x,y,z,t)=\frac{1}{zx^{13}}\left(831600-415800t^2+34650t^4-831600\cos(t)\right)+\frac{1}{x^5y^9}\left(1680-840t^2+70t^4-1680\cos(t)\right)+\frac{24z\cos(t)}{x^5}$$
$$\frac{\partial^4}{\partial y^4}u_2(x,y,z,t)=\frac{1}{xy^{13}}\left(831600-415800t^2+34650t^4-831600\cos(t)\right)+\frac{1}{y^5z^9}\left(1680-840t^2+70t^4-1680\cos(t)\right)+\frac{24x\cos(t)}{y^5} \tag{89}$$
$$\frac{\partial^4}{\partial z^4}u_2(x,y,z,t)=\frac{1}{yz^{13}}\left(831600-415800t^2+34650t^4-831600\cos(t)\right)+\frac{1}{x^9z^5}\left(1680-840t^2+70t^4-1680\cos(t)\right)+\frac{24y\cos(t)}{z^5}$$

$$\frac{1}{4!z}\frac{\partial^4}{\partial x^4}u_2(x,y,z,t)=\frac{34650}{z^2x^{13}}\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\cos(t)\right)+\frac{70}{x^5y^9z}\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\cos(t)\right)+\frac{\cos(t)}{x^5} \tag{89}$$

$$\frac{1}{4!x}\frac{\partial^4}{\partial y^4}u_2(x,y,z,t)=\frac{34650}{x^2y^{13}}\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\cos(t)\right)+\frac{70}{xy^5z^9}\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\cos(t)\right)+\frac{\cos(t)}{y^5}$$

$$\frac{1}{4!y}\frac{\partial^4}{\partial z^4}u_2(x,y,z,t)=\frac{34650}{y^2z^{13}}\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\cos(t)\right)+\frac{70}{x^9yz^5}\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\cos(t)\right)+\frac{\cos(t)}{z^5} \tag{90}$$

Using the Maple package, the inverse Fourier Transform, $u_3(x,y,z,t)$ is:

$$u_3(x,y,z,t)=\frac{1}{72}\frac{1}{x^{13}y^{13}z^{13}}\begin{bmatrix} 5040\cos(t)\left(z^{12}x^8y^4+x^{12}y^8z^4+y^{12}z^8x^4\right) \\ -210t^4\left(z^{12}x^8y^4+x^{12}y^8z^4+y^{12}z^8x^4\right)+2520t^2\left(z^{12}x^8y^4+x^{12}y^8z^4+y^{12}z^8x^4\right) \\ 3465t^6\left(z^{11}y^{13}+z^{13}x^{11}+y^{11}x^{13}\right)-2494800\left(z^{11}y^{13}+z^{13}x^{11}+y^{11}x^{13}\right) \\ +1247400t^2\left(z^{11}y^{13}+z^{13}x^{11}+y^{11}x^{13}\right)-103950t^4\left(z^{11}y^{13}+z^{13}x^{11}+y^{11}x^{13}\right) \\ -5040\left(z^{12}x^8y^4+x^{12}y^8z^4+y^{12}z^8x^4\right)+2494800\cos(t)\left(z^{11}y^{13}+z^{13}x^{11}+y^{11}x^{13}\right) \\ +7t^6\left(z^{12}x^8y^4+x^{12}y^8z^4+y^{12}z^8x^4\right)+72\cos(t)\left(z^{13}y^{12}x^{14}+z^{14}y^{13}x^{12}+z^{12}y^{14}x^{13}\right) \end{bmatrix}$$

$$\rightarrow u_3(x,y,z,t)=\frac{1}{72}\frac{1}{x^{13}y^{13}z^{13}}\begin{bmatrix} 5040\left(z^{12}x^8y^4+x^{12}y^8z^4+y^{12}z^8x^4\right)\left(\cos(t)-1+\frac{t^2}{2!}-\frac{t^4}{4!}+\frac{t^6}{6!}\right) \\ +2494800\left(z^{11}y^{13}+z^{13}x^{11}+y^{11}x^{13}\right)\left(\cos(t)-1+\frac{t^2}{2!}-\frac{t^4}{4!}+\frac{t^6}{6!}\right) \\ +72\cos(t)\left(z^{13}y^{12}x^{14}+z^{14}y^{13}x^{12}+z^{12}y^{14}x^{13}\right) \end{bmatrix}$$

$$\rightarrow u_3(x,y,z,t)=\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+34650\left(\frac{1}{z^{13}y^2}+\frac{1}{y^{13}x^2}+\frac{1}{z^2x^{13}}\right)+70\left(\frac{1}{z^5yx^9}+\frac{1}{x^5zy^9}+\frac{1}{y^5xz^9}\right)\right]\cos(t)-\left[34650\left(\frac{1}{z^{13}y^2}+\frac{1}{y^{13}x^2}+\frac{1}{z^2x^{13}}\right)\right.$$

$$\left.+70\left(\frac{1}{z^5yx^9}+\frac{1}{x^5zy^9}+\frac{1}{y^5xz^9}\right)\right]\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}\right) \tag{91}$$

And so on. The Taylor series expansion for cos(t) is written as:

$$\cos(t)=\sum_{i=0}^{n}\frac{(-1)^i t^{2i}}{(2i)!}=1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots \tag{92}$$

And

$$u(x,t)=\lim_{n\to\infty}u_n(x,t) \tag{93}$$

By substituting Eq. (91) and Eq. (92) into Eq. (93) thus Eq. (91) can ultimately be reduced to Eq (94):

$$u_3(x,y,z,t)=\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+34650\left(\frac{1}{z^{13}y^2}+\frac{1}{y^{13}x^2}+\frac{1}{z^2x^{13}}\right)+70\left(\frac{1}{z^5yx^9}+\frac{1}{x^5zy^9}+\frac{1}{y^5xz^9}\right)\right]\cos(t)-$$

$$\begin{bmatrix} 34650\left(\frac{1}{z^{13}y^2}+\frac{1}{y^{13}x^2}+\frac{1}{z^2x^{13}}\right)+ \\ 70\left(\frac{1}{z^5yx^9}+\frac{1}{x^5zy^9}+\frac{1}{y^5xz^9}\right) \end{bmatrix}\cos(t) \tag{94}$$

Therefore, the exact solution is given as:

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\cos(t) \tag{95}$$

- **EXAMPLE 4**

Consider the following singular fourth parabolic partial differential.

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u}{\partial x^4}=0 \tag{96}$$

With initial conditions:

$$u(x,0)=x-\sin(x) \qquad 0<x<1 \tag{97}$$

$$\frac{\partial u}{\partial t}(x,0)=-[x-\sin(x)] \qquad 0<x<1 \tag{98}$$

Now using FTVIM we construct a correction functional as:

$$\hat{u}_{n+1}(\omega,t)=\hat{u}_n(\omega,t)+\int_0^t \lambda(\xi)\times F\left\{\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}+\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right\}d\xi \tag{99}$$

Integrating by parts and take first variation, we get $\lambda(\xi)=\xi-t$ as follows:

$$\Rightarrow \delta\hat{u}_{n+1}=\delta\hat{u}_n+\delta\int_0^t \lambda(\xi)\times\left[\frac{\partial^2\hat{u}_n}{\partial\xi^2}+F\left\{\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_n}{\partial x^4}\right\}\right]d\xi \tag{100}$$

$$\Rightarrow \delta\hat{u}_{n+1}=\delta\hat{u}_n+\delta\int_0^t \lambda(\xi)\times\frac{\partial\hat{u}_n}{\partial\xi^2}d\xi \tag{101}$$

$$\Rightarrow \delta\hat{u}_{n+1}=\delta\hat{u}_n+\delta\left[\lambda(\xi)\times\frac{\partial\hat{u}_n}{\partial\xi}-\lambda'(\xi)\times\hat{u}_n+\int_0^t\lambda''(\xi)\times\hat{u}_n\, d\xi\right] \tag{102}$$

$$\Rightarrow \delta\hat{u}_{n+1}=\delta\hat{u}_n+\delta\left[\lambda(\xi)\times\frac{\partial\hat{u}_n}{\partial\xi}-\lambda'(\xi)\times\hat{u}_n+\int_0^t\lambda''(\xi)\times\hat{u}_n\, d\xi\right] \tag{103}$$

$$\Rightarrow\begin{cases}1-\lambda'(\xi)\big|_{\xi=t}=0\\ \lambda(\xi)\big|_{\xi=t}=0 \qquad \Rightarrow \lambda(\xi)=\xi-t\\ \lambda''(\xi)\big|_{\xi=t}=0\end{cases} \tag{104}$$

Assuming $u_0(x,t) = (x - \sin(x))(1-t)$, using the same method as used in [3], and substituting for the value of $\lambda(\xi)=\xi-t$, into Eq. (99) the successive approximations, $\hat{u}_{n+1}(\omega,t)$, are obtained as follows:

$$\hat{u}_1(\omega,t)=\hat{u}_0(\omega,t)+\int_0^t\lambda(\xi)\times\left[\frac{\partial^2\hat{u}_0(\omega,\xi)}{\partial\xi^2}+F\left\{\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_0(x,\xi)}{\partial x^4}\right\}\right]d\xi \tag{105}$$

$$\hat{u}_2(\omega,t)=\hat{u}_1(\omega,t)+\int_0^t\lambda(\xi)\times\left[\frac{\partial^2\hat{u}_1(\omega,\xi)}{\partial\xi^2}+F\left\{\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_1(x,\xi)}{\partial x^4}\right\}\right]d\xi \tag{106}$$

$$\hat{u}_3(\omega,t)=\hat{u}_2(\omega,t)+\int_0^t\lambda(\xi)\times\left[\frac{\partial^2\hat{u}_2(\omega,\xi)}{\partial\xi^2}+F\left\{\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_2(x,\xi)}{\partial x^4}\right\}\right]d\xi \tag{107}$$

$$\hat{u}_{n+1}(\omega,t)=\hat{u}_n(\omega,t)+\int_0^t\lambda(\xi)\times\left[\frac{\partial^2\hat{u}_n(\omega,\xi)}{\partial\xi^2}+F\left\{\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right\}\right]d\xi \tag{108}$$

For obtaining $\hat{u}_1(\omega, t)$, first we calculate followings:

$$\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_0}{\partial x^4}=(\sin(x)-x)\times(1-t) \tag{109}$$

$$F\{(\sin(x)-x)\times(1-t)\}=\left(\frac{1}{2}I\pi Dirac(\omega+1)-\frac{1}{2}I\pi Dirac(\omega-1)-\frac{1}{(\omega+z)(\omega-1)}-\frac{I\left(\omega^2\pi Dirac(1,\omega)+I\right)}{\omega^2}\right)\times(1-t) \tag{110}$$

Using Eq. (105) we obtain $\hat{u}_1(\omega, t)$ as:

$$\hat{u}_1(\omega,t)=\left(\frac{I\left(\omega^2\pi Dirac(1,\omega)+I\right)}{\omega^2}-\frac{1}{2}I\pi Dirac(\omega+1)+\frac{1}{2}I\pi Dirac(\omega-1)-\frac{1}{(\omega+z)(\omega-1)}\right)\times(1-t)+\int_0^t(\xi-t)\times\left(\frac{1}{2}I\pi Dirac(\omega+1)-\frac{1}{2}I\pi Dirac(\omega-1)\right.$$
$$\left.-\frac{1}{(\omega+z)(\omega-1)}-\frac{I\left(\omega^2\pi Dirac(1,\omega)+I\right)}{\omega^2}\right)\times(1-\xi)d\xi\Rightarrow\hat{u}_1=\left(\frac{I\left(\omega^2\pi Dirac(1,\omega)+I\right)}{\omega^2}-\frac{1}{2}I\pi Dirac(\omega+1)+\frac{1}{2}I\pi Dirac(\omega-1)-\frac{1}{(\omega+z)(\omega-1)}\right)\times$$
$$(1-t)+\left(\frac{1}{2}I\pi Dirac(\omega+1)-\frac{1}{2}I\pi Dirac(\omega-1)-\frac{I\left(\omega^2\pi Dirac(1,\omega)+I\right)}{\omega^2}-\frac{1}{(\omega+z)(\omega-1)}\right)\times\left(-\frac{1}{3}t^3+\frac{1}{2}(t+1)t^2-t^2\right) \tag{111}$$

Using the Maple package, the inverse Fourier Transform, $u_1(x,t)$ is:

$$u_1(x,t)=\left(x-\sin(x)\right)\times\left(1-t+\frac{t^2}{2}-\frac{t^3}{6}\right) \tag{112}$$

For $u_2(x,t)$

$$\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_1}{\partial x^4}=\left(\sin(x)-x\right)\times\left(1-t+\frac{t^2}{2}-\frac{t^3}{6}\right) \tag{113}$$

Using correction functional and the Maple package, the inverse Fourier transform, $u_2(x,t)$ is:

$$u_2(x,t)=\left(x-\sin(x)\right)\times\left(1-t+\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}-\frac{t^5}{120}\right) \tag{114}$$

For $u_3(x,t)$

$$\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_2}{\partial x^4}=\left(\sin(x)-x\right)\times\left(1-t+\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}-\frac{t^5}{120}\right) \tag{115}$$

Using correction functional and the Maple package, the inverse Fourier transform, $u_3(x,t)$ is:

$$u_3(x,t)=\left(x-\sin(x)\right)\times\left(1-t+\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}-\frac{t^5}{120}+\frac{t^6}{720}-\frac{t^7}{5040}\right) \tag{116}$$

And so on. The Taylor series expansion for e^{-t} is written as:

$$e^{-t}=\sum_{i=0}^{n}\frac{(-1)^i t^i}{i!}=1-t+\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}-\frac{t^5}{120}+\frac{t^6}{720}-\frac{t^7}{5040}+\ldots \tag{117}$$

And:

$$u(x,t)=\lim_{n\to\infty}u_n(x,t) \tag{118}$$

By substituting Eq. (116) and Eq. (117) into Eq. (118) thus Eq. (116) can ultimately be reduced to:

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\left(x-\sin(x)\right)\times e^{-t} \tag{119}$$

Which, it is the exact solution of Eq. (96).

RESULTS

In the following tables and figures, we show the trend of convergence of u_0 to u_3 of the FTVIM and MVIM solution at three different locations and at three different times. The trend of rapid convergence of the FTVIM towards the exact solution is clearly shown. Using the new method, FTVIM, indicates that the amount of computational work is much less than that of the MVIM.

Following tables below exhibits the relative errors obtained by the FTVIM , MVIM and the exact solution.

Table 1

indicates the relative errors of the results of the u_0 to u_3 of the FTVIM and MVIM for example 1

		relative error					
		x =1		x =1.5		x =2	
		FTVIM	MVIM	FTVIM	MVIM	FTVIM	MVIM
t =0.1	$u_0(x, t)$	0.001668613163482	1.000000000000000	0.001668613163480	1.000000000000000	0.001668613163475	1.000000000000000
	$u_1(x, t)$	8.345251287027870 e - 7	0.001668613163482	8.345251289509500 e - 7	0.001668613163480	8.345251299445210 e - 7	0.001668613163475
	$u_2(x, t)$	1.987178076468330 e - 10	0.001668613163482	1.987169220379880 e - 10	0.001668613163480	1.987099650913570 e - 10	0.001668613163475
	$u_3(x, t)$	1.9867 80720324260 e - 14	8.345251287027870 e - 7	2.826162729278290 e - 14	8.345251289509500 e - 7	3.163164041568870 e - 14	8.345251299445210 e - 7
t =0.4	$u_0(x, t)$	0.027172982219114	1.000000000000000	0.027172982219115	1.000000000000000	0.027172982219115	1.000000000000000
	$u_1(x, t)$	0.000218297306730	0.027172982219114	0.000218297306729	0.027172982219115	0.000218297306730	0.027172982219115
	$u_2(x, t)$	8.329294753150260 e - 7	0.027172982219114	8.329294772280170 e - 7	0.027172982219115	8.329294766386410 e - 7	0.027172982219115
	$u_3(x, t)$	1.852374893422060 e - 9	0.000218297306730	1.852375831519450 e - 9	0.000218297306729	1.852376049494290 e - 9	0.000218297306730
t =0.7	$u_0(x, t)$	0.086589228869972	1.000000000000000	0.086589228869973	1.000000000000000	0.086589228869973	1.000000000000000
	$u_1(x, t)$	0.002148891487743	0.086589228869972	0.002148891487742	0.086589228869973	0.002148891487742	0.086589228869973
	$u_2(x, t)$	0.000025192461021	0.086589228869972	0.000025192461021	0.086589228869973	0.000025192461022	0.086589228869973
	$u_3(x, t)$	1.718517149922930 e - 7	0.002148891487743	1.718517147739880 e - 7	0.002148891487742	1.718517135034890 e - 7	0.002148891487742

Table 2

indicates the relative errors of the results of the u_0 to u_3 of the FTVIM and MVIM for example 2

		relative error					
		(x, y) =(1, 1)		(x, y) =(1.5, 1.5)		(x, y) =(2, 2)	
		FTVIM	MVIM	FTVIM	MVIM	FTVIM	MVIM
t =0.1	$u_0(x, y, t)$	0.001668613163476	1.000000000000000	0.001668613163479	1.000000000000000	0.001668613163476	1.000000000000000
	$u_1(x, y, t)$	8.345251280689990 e - 7	0.001668613163476	8.345251247541810 e - 7	0.001668613163479	8.345251299445220 e - 7	0.001668613163476
	$u_2(x, y, t)$	1.987154929508460 e - 10	0.001668613163476	1.987174944070330 e - 10	0.001668613163479	1.987121438012830 e - 10	0.001668613163476
	$u_3(x, y, t)$	3.000838008922480 e - 14	8.345251280689990 e - 7	2.465171745528260 e - 14	8.345251247541810 e - 7	3.219649113739740 e - 14	8.345251299445220 e - 7
t =0.4	$u_0(x, y, t)$	0.027172982219114	1.000000000000000	0.027172982219114	1.000000000000000	0.027172982219114	1.000000000000000
	$u_1(x, y, t)$	0.000218297306730	0.027172982219114	0.000218297306730	0.027172982219114	0.000218297306729	0.027172982219114
	$u_2(x, y, t)$	8.329294758766400 e - 7	0.027172982219114	8.329294760610980 e - 7	0.027172982219114	8.329294761628440 e - 7	0.027172982219114
	$u_3(x, y, t)$	1.852375832984920 e - 9	0.000218297306730	1.852375593306180 e - 9	0.000218297306730	1.852375718504080 e - 9	0.000218297306729
t =0.7	$u_0(x, y, t)$	0.086589228869979	1.000000000000000	0.086589228869971	1.000000000000000	0.086589228869976	1.000000000000000
	$u_1(x, y, t)$	0.002148891487736	0.086589228869979	0.002148891487741	0.086589228869971	0.002148891487742	0.086589228869976
	$u_2(x, y, t)$	0.000025192461024	0.086589228869979	0.000025192461017	0.086589228869971	0.000025192461023	0.086589228869976
	$u_3(x, y, t)$	1.718517101092090 e - 7	0.002148891487736	1.718517152635170 e - 7	0.002148891487741	1.718517140537040 e - 7	0.002148891487742

Table 3

indicates the relative errors of the results of the u_0 to u_3 of the FTVIM and MVIM for example 3

		relative error					
		(x, y, z) =(1, 1, 1)		(x, y, z) =(1.5, 1.5, 1.5)		(x, y, z) =(2, 2, 2)	
		FTVIM	MVIM	FTVIM	MVIM	FTVIM	MVIM
t =0.1	$u_0(x, y, z, t)$	0.005020918400455	0.005020918400455	0.005020918400455	0.005020918400455	0.005020918400455	0.005020918400455
	$u_1(x, y, z, t)$	0.000004186191546	0.005020918400455	5.512680226302784 e - 7	0.005020918400455	1.308184858390211 e - 7	0.005020918400455
	$u_2(x, y, z, t)$	9.769292072542849 e - 8	0.000004186191546	1.694144707325082 e - 9	5.512680226302784 e - 7	9.54032428959262 e - 11	1.308184858390211 e - 7
	$u_3(x, y, z, t)$	0.085704428383238	0.000004186191546	1.97613747672213 e - 11	5.512680226302784 e - 7	2.64080361208883 e - 13	1.308184858390211 e - 7
t =0.4	$u_0(x, y, z, t)$	0.001151925887420	0.085704428383238	0.085704428383238	0.085704428383238	0.085704428383238	0.085704428383238
	$u_1(x, y, z, t)$	0.000431118533187	0.085704428383238	0.000151693944022	0.085704428383238	0.000035997683981	0.085704428383238
	$u_2(x, y, z, t)$	0.000611616091983	0.001151925887420	0.000007476254940	0.000151693944022	4.210141925660517 e - 7	0.000035997683981
	$u_3(x, y, z, t)$	1.852374893422060 e - 9	0.001151925887420	0.000001396722140	0.000151693944022	1.866504186960827 e - 8	0.000035997683981
t =0.7	$u_0(x, y, z, t)$	0.307459259733593	0.307459259733593	0.307459259733593	0.307459259733593	0.307459259733593	0.307459259733593
	$u_1(x, y, z, t)$	0.012868258901136	0.307459259733593	0.001694585534306	0.307459259733593	0.000402133090660	0.307459259733593
	$u_2(x, y, z, t)$	0.014824701018040	0.012868258901136	0.000257082996197	0.001694585534306	0.000014477247087	0.000402133090660
	$u_3(x, y, z, t)$	0.064551974007875	0.012868258901136	0.000147414648676	0.001694585534306	0.000001969969909	0.000402133090660

Table 4

indicates the relative errors obtained by the FTVIM, MVIM and the exact solution for example 4

		relative error					
		x =1		x =1.5		x =2	
		FTVIM	MVIM	FTVIM	MVIM	FTVIM	MVIM
t =0.1	$u_0(x, t)$	0.005346173731916	0.105170918075647	0.005346173731918	0.105170918075647	0.005346173731917	0.105170918075647
	$u_1(x, t)$	0.000004514294554	0.005346173731916	0.000004514294552	0.005346173731918	0.000004514294551	0.005346173731917
	$u_2(x, t)$	1.513305096916960 e - 9	0.000004514294554	1.513301488251290 e - 9	0.000004514294552	1.513302398108410 e - 9	0.000004514294551
	$u_3(x, t)$	2.718850284455520 e - 13	1.513305096916960 e - 9	2.705167496830410 e - 13	1.513301488251290 e - 9	2.70541797905007 e - 13	1.513302398108410 e - 9
t =0.4	$u_0(x, t)$	0.104905181415237	0.491824697641270	0.104905181415237	0.491824697641270	0.104905181415237	0.491824697641270
	$u_1(x, t)$	0.001472002378775	0.104905181415237	0.001472002378776	0.104905181415237	0.001472002378777	0.104905181415237
	$u_2(x, t)$	0.000008025075486	0.001472002378775	0.000008025075489	0.001472002378776	0.000008025075490	0.001472002378777
	$u_3(x, t)$	2.321196910569100 e - 8	0.000008025075486	2.321196991935650 e - 8	0.000008025075489	2.321196884804210 e - 8	0.000008025075490
t =0.7	$u_0(x, t)$	0.395874187758858	1.013752707470476	0.395874187758856	1.013752707470476	0.395874187758858	1.013752707470476
	$u_1(x, t)$	0.017624304205654	0.395874187758858	0.017624304205655	0.395874187758856	0.017624304205654	0.395874187758858
	$u_2(x, t)$	0.000298814974205	0.017624304205654	0.000298814974202	0.017624304205655	0.000298814974205	0.017624304205654
	$u_3(x, t)$	0.000002669983854	0.000298814974205	0.000002669983852	0.000298814974202	0.000002669983854	0.000298814974205

The efficiency and rapid convergence of FTVIM are shown in the following figures:

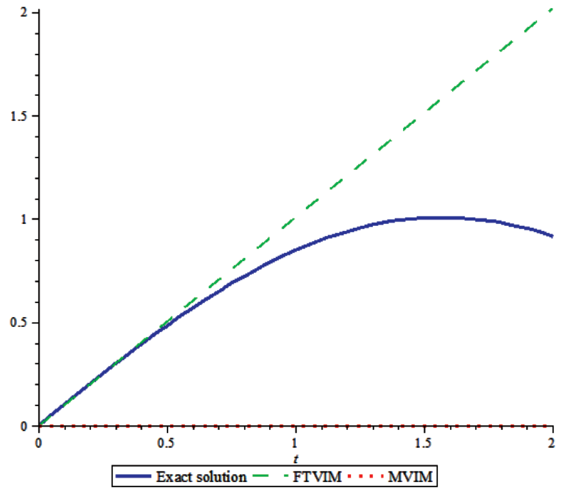

Figure 1: exact solution and first approximant of $u_0(x, t)$ for x=1 in example 1

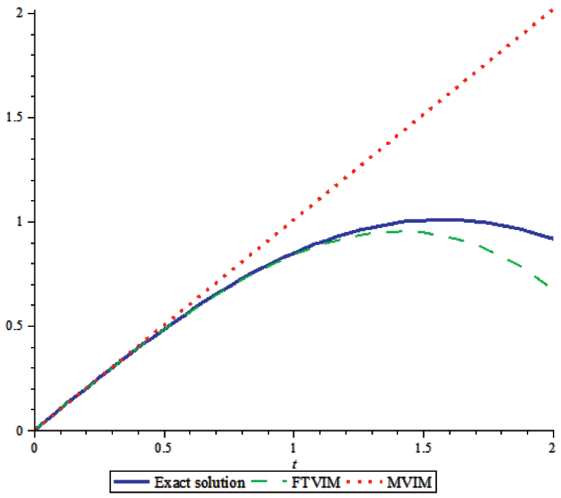

Figure 2: exact solution and second approximant of u1 (x, t) for x=1 in example 1

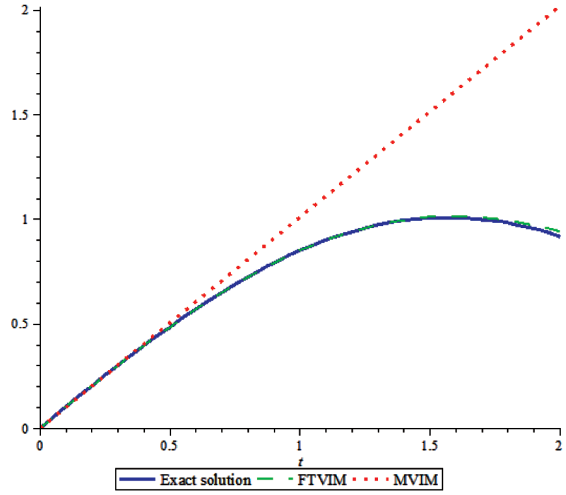

Figure 3: exact solution and third approximant of $u_2(x, t)$ for x=1 in example 1

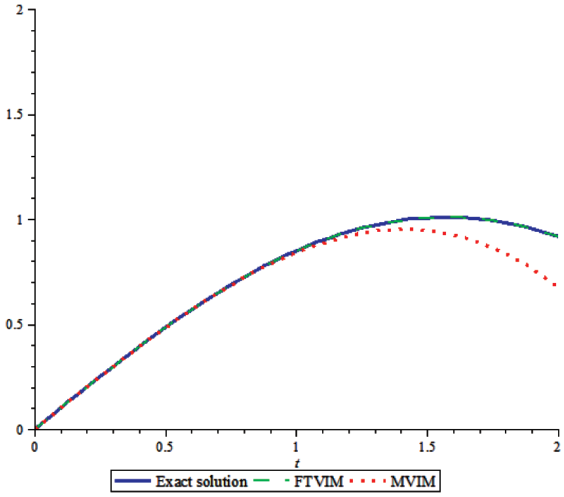

Figure 4: exact solution and fourth approximant of u3(x, t) for x=1 in example 1

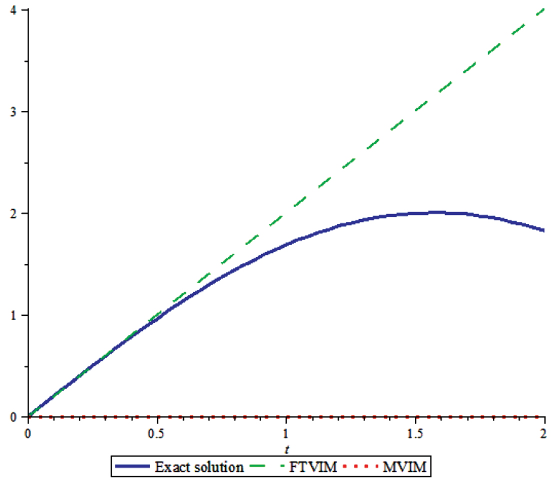

Figure 5: exact solution and first approximant of $u_0(x, y, t)$ for x=1, y=1 in example 2

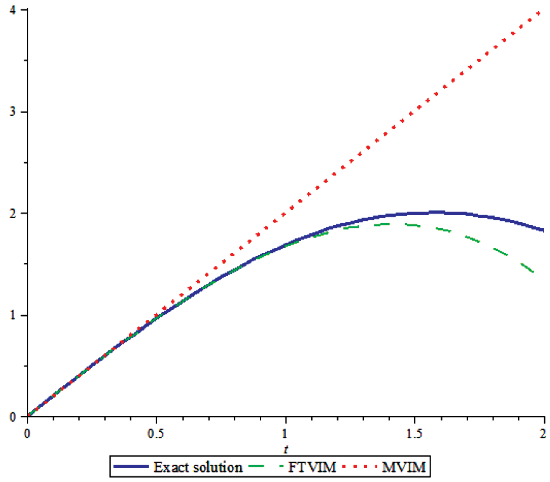

Figure 6: exact solution and second approximant of u1 (x, y, t) for x=1, y=1 in example 2

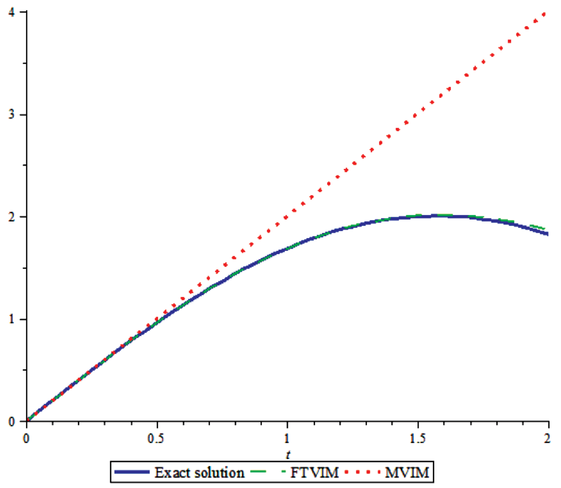

Figure 7: exact solution and third approximant of $u_2(x, y, t)$ for x=1, y=1 in example 2

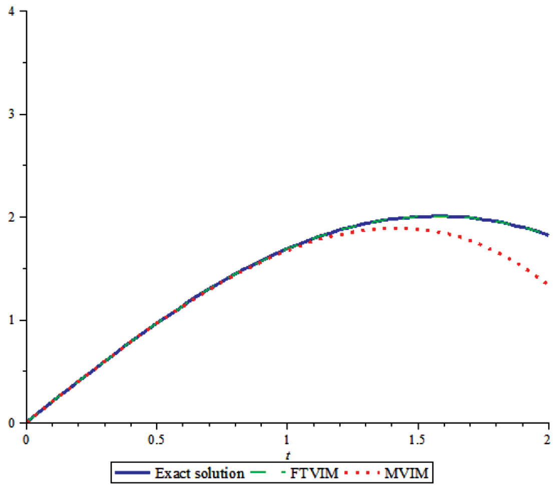

Figure 8: exact solution and fourth approximant of $u_3(x, y, t)$ for x=1, y=1 in example 2

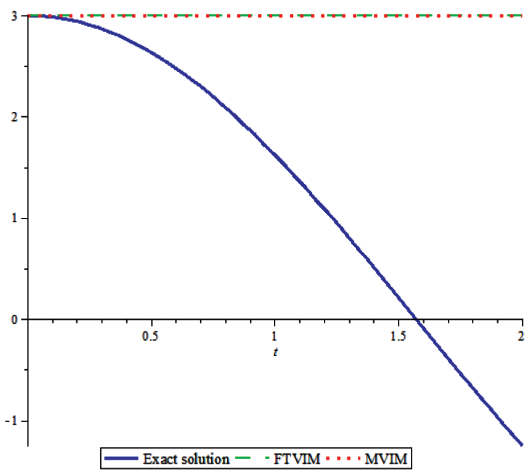

Figure 9: exact solution and first approximant of $u_0(x, y, z, t)$ for x=2, y=2, z=2 in example 3

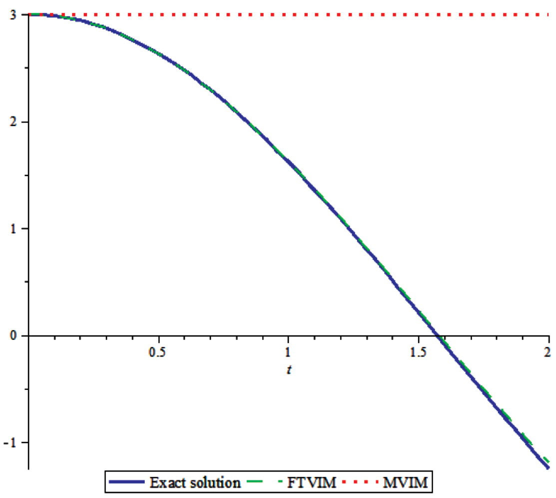

Figure 10: exact solution and second approximant of u1 (x, y, z, t) for x=2, y=2, z=2 in example 3

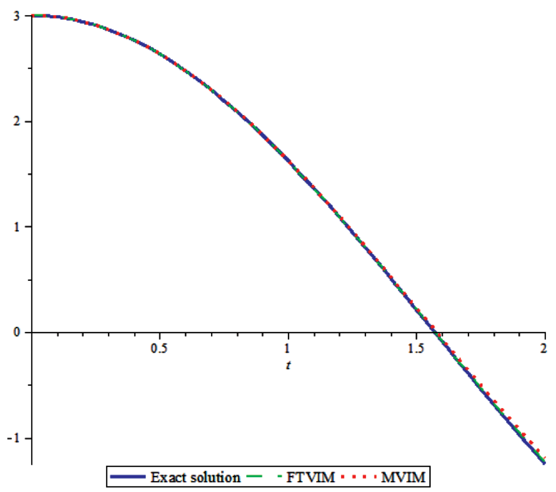

Figure 11: exact solution and third approximant of $u_2(x, y, z, t)$ for x=2, y=2, z=2 in example 3

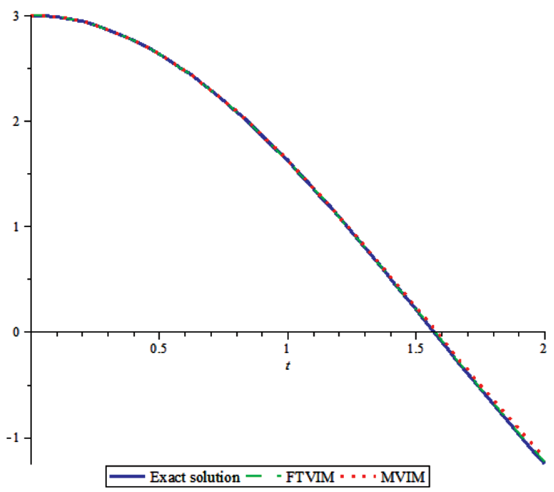

Figure 12: exact solution and fourth approximant of $u_3(x, y, z, t)$ for x=2, y=2, z=2 in example 3

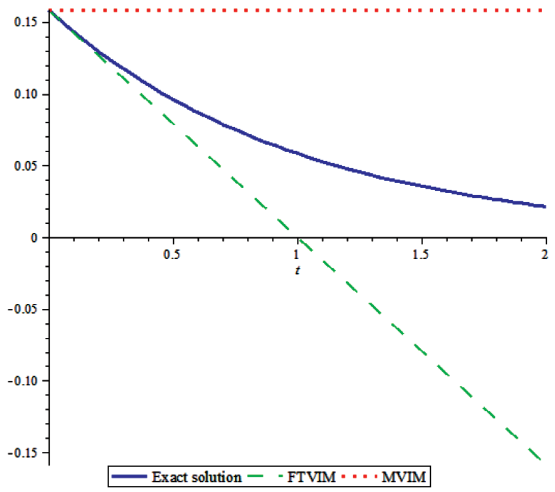

Figure 13: exact solution and first approximant of $u_0(x, t)$ for x=1 in example 4

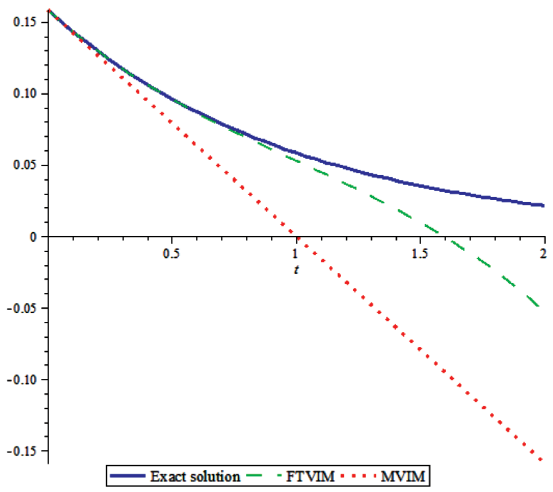

Figure 14: exact solution and second approximant of u1 (x, t) for x=1 in example 4

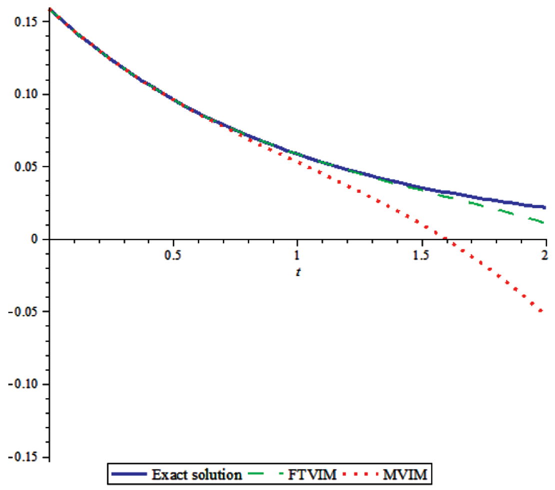

Figure 15: exact solution and third approximant of u2(x, t) for x=1 in example 4

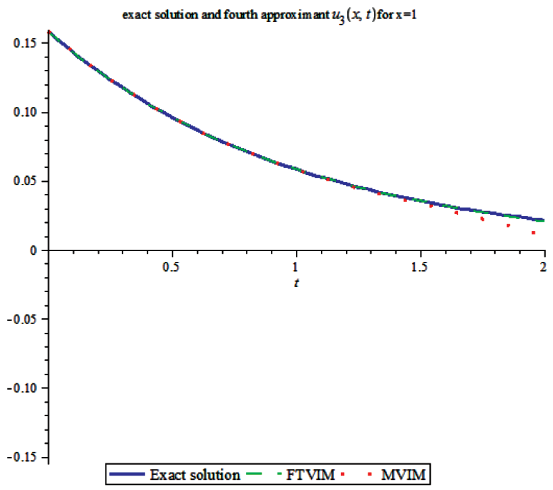

Figure 16: exact solution and fourth approximant of u3(x, t) for x=1 in example 4

3- CONCLUSIONS

A new effective modification to the variational iteration method, the Fourier transform variational iteration method (FTVIM), is presented in this paper. The results obtained by FTVIM are compared with MVIM. The validity and effectiveness of the new method, FTVIM is shown by solving four singular differential equations with variable coefficients and the very rapid approach to the exact solutions is shown schematically. The very rapid approach towards the exact solutions of the new method, FTVIM, indicates that the amount of computational work is much less than those required for that of MVIM. Moreover, the deficiency of the MVIM caused by unsatisfied boundary conditions is overcome by the new method, FTVIM, where, the solution is shown to be valid in the entire range of problem domain. It is concluded that the FTVIM is a powerful and efficient tool in obtaining the accurate solutions as well as other effective numerical methods.

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