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Conservative chaotic flow generated via a pseudo-linear system

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ABSTRACT: Analysis of nonlinear autonomous systems has been a popular field of study in recent decades. As a noticeable nonlinear behavior, chaotic dynamics has been intensively investigated since Lorenz discovered the first physical evidence of chaos in his famous equations. Although many chaotic systems have been ever reported in the literature, a systematic and qualitative approach for chaos generation is still a challenging issue. Recently, we have developed an analysis tool which provides globally valid results about the qualitative behavior of some nonlinear systems based on their pseudo-linear form of representation. In this paper, it is applied to generate conservative chaos by focusing on the essential qualitative attribute of conservative chaotic behavior. This feature is the continual stretching and folding of system trajectories which never settle down to a periodic regime. Indeed, it is tried to create this quality of behavior through the aforementioned qualitative analysis tool. The proposed approach helps us to generate a new class of chaotic systems with highly remarkable characteristics. The most elegant one is its almost parameter independency for chaos generation; There is no need for a trial-and-error mechanism to find the exact parameters' values in order to produce chaotic behavior. It is shown that the system exhibits conservative chaotic dynamics for almost all parameters' values. The chaotic behavior of the derived system is verified through the analysis of Lyapunov exponents and dimension as well. Besides, the frequency power spectrum analysis is also performed in order to put more emphasis on the chaotic behavior of the system.

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1. Introduction

As an interesting nonlinear behavior, chaotic dynamics has been intensively investigated since Lorenz discovered the first physical evidence of chaos in Lorenz, 1963. Afterward, various new chaotic systems have been reported in the literature. Although the main research in this area was the analysis and experimental confirmation of chaotic systems, other research areas such as chaos control, synchronization of chaotic systems and chaos generation were also explored. Similar to the mainstream research of controlling or suppressing chaos, the opposite direction of making a non-chaotic dynamical system chaotic, i.e., chaos generation has also recently attracted increasing attention especially in secure communications, resonance prevention in mechanical systems, material damage control in lasers, fluid mixing and nonlinear optics. In particular, within the biological context, Raiesdana and Goplayegani, have shown in [1] that anti-control of chaos or chaotification has a great potential application in seizures control in an epileptic brain. A drawback in this research field is the existence of the aforementioned chaotic attractors usually indicated by numerical simulations or it is confirmed by physical circuits or by computer-assisted proofs. Although Shil'nikov theorem [2] has provided a rigorous theoretical tool for 3-dimensional continuous autonomous systems, the existence of chaotic attractors is still tough to prove since it is hard to satisfy the assumption that there exist homoclinic orbits or heteroclinic orbits. It seems that except for some exceptional cases, in general, there is no analytical method yet easily applicable for chaos generation in smooth autonomous nonlinear systems.

Motivated by the need for making chaos more suitable for real-world applications, this paper introduces an elegant analytical mechanism for generating chaos based on the concept of Eigen-structure analysis of nonlinear systems. The Eigen-structure analysis of nonlinear system based on their pseudo-linear (PL) form are fully described and presented in [3] and [4]. In addition, a dissipative chaotic flow and its dynamical analysis have been presented in [14], based on this method.

In this paper, we apply this PL form of representation for the synthesis of essential qualitative characteristics of a conservative chaotic system. Among these qualitative characteristics, we focus on the generation of a dynamical system with continual stretching and folding trajectories. This trend provides a qualitative and almost parameter independent chaos generation scheme. The rest of the paper is organized as follows. In section 2, the PL representation of

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nonlinear systems is briefly introduced and the ability of this representation for the determination of qualitative behavior of nonlinear systems is discussed. Then, in section 3, the basic idea of this paper for chaos generation based on the qualitative analysis of nonlinear systems is presented. Section 4 is devoted to analyzing the numerical simulation results, which highlight the strength of the proposed approach. Finally, concluding remarks and future work are presented in section 5.

2. Pseudo Linear systems; A brief review([3]-[4])

An autonomous nonlinear system is a system of nonlinear ordinary differential equations which does not explicitly depend on the independent variable. It is of the form:

$$\dot{\mathbf{x}}(t)=\mathbf{f}\left(\mathbf{x}(t)\right) \tag{2.1}$$

where $\mathbf{x}$ takes values in n - dimensional Euclidian space and the independent variable t is usually time. Inspiring from the linear system theory, assuming that $\mathbf{f}(\mathbf{0})=\mathbf{0}$, it is possible to transform an autonomous system (2.1) to a new form as:

$$\frac{d}{dt}\mathbf{x}(t)=A\left(\mathbf{x}(t)\right)\mathbf{x}(t) \tag{2.2}$$

where $A:\mathbb{R}^n\to\mathbb{R}^{n\times n}$. This form is called pseudo-linear (PL) and it was initially introduced in [5] to cope with the difficulty of designing nonlinear optimal control laws. After obtaining the PL form of (2.2), it would be possible to extend the Eigen-structure concept to these nonlinear systems. In other words, by defining $\lambda(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ as nonlinear eigenvalue (NEValue) and its corresponding nonlinear eigenvector (NEVector), one can write:

$$A(\mathbf{x})\mathbf{v}(\mathbf{x})=\lambda(\mathbf{x})\mathbf{v}(\mathbf{x}) \tag{2.3}$$

Then, the nonlinear eigenvalues are achieved as the solution of the following equation:

$$\left|A(\mathbf{x})-\lambda(\mathbf{x})I_n\right|=0 \tag{2.4}$$

Based on these introductory definitions, some fruitful propositions, corollaries, comments and definitions are presented in the following. Applying these results, the qualitative analysis of some nonlinear dynamical systems is attainable. The proofs of all of them have been given in [3] and [4].

PROPOSITION 2.1 Consider a nonlinear system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ where $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ with $\mathbf{f}(\mathbf{0})=\mathbf{0}$. This system may have infinite distinct PL forms. However, there are at most m possible PL forms as a basis set for these infinite PL forms in such a way that every PL form system matrix of this system can be obtained as a convex linear combination of this basis set. The number m is indeed calculated by

$$m=\prod_{i=1}^{n}m_i=\prod_{i=1}^{n}\left(n^{k_i}\right)=n^{\sum_{i=1}^{n}k_i} \tag{2.4}$$

where k_i stands for the number of distinct terms in $f_i(\mathbf{x})$. Then, the infinite PL form system matrices are obtained from the basis set $\left\{A_1(\mathbf{x}),A_2(\mathbf{x}),\ldots,A_m(\mathbf{x})\right\}$ as follows:

$$B_j(\mathbf{x})=\sum_{i=1}^{m}\alpha_{ji}A_i(\mathbf{x}) \quad ; \quad \sum_{i=1}^{m}\alpha_{ji}=1, \quad j=1,2,\ldots \tag{2.5}$$

For nonlinear system analysis, only the m PL forms with system matrices of $\left\{A_1(\mathbf{x}),A_2(\mathbf{x}),\ldots,A_m(\mathbf{x})\right\}$ are needed to be considered. Among these m PL forms, only the PL form which has state independent (SI) NEVectors must be used in Eigen-structure based analysis of the system. The reason is only a PL form with SI NEVectors certainly leads to correct qualitative results through its NEValues analysis.

PROOF: See [3].

In the light of the above proposition which clarifies the conditions for the correct determination of the qualitative behavior of nonlinear systems based on their PL forms, the following results are dedicated to the stability analysis of nonlinear systems, which is one of the key points of qualitative behavior.

PROPOSITION 2.2 For a nonlinear system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ where $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ with $\mathbf{f}(\mathbf{0})=\mathbf{0}$, a sufficient condition for global asymptotic stability of the origin is that the system has a PL form representation, $\dot{\mathbf{x}}=A(\mathbf{x})\mathbf{x}$, which satisfies the following conditions:

1. $\forall\mathbf{x}\in\mathbb{R}^n \mid \mathrm{Re}\left\{\lambda_i(\mathbf{x})\right\}<0;\quad i=1,2,\ldots,n$.
2. The geometric multiplicity of every multiple NEValue equals to its corresponding algebraic multiplicity.
3. All NEVectors of the matrix $A(\mathbf{x})$ are SI.

PROOF: See [4].

COROLLARY 2.1. Consider the system defined in Proposition 2.2. Then, sufficient conditions for instability of the origin are:

1. $\forall\mathbf{x}\in\mathbb{R}^n \mid \mathrm{Re}\left\{\lambda_i(\mathbf{x})\right\}>0;\quad i=1,2,\ldots,n$.
2. The geometric multiplicity of every multiple NEValue equals to its corresponding algebraic multiplicity.
3. All NEVectors of the matrix $A(\mathbf{x})$ should be SI.

PROOF: See [4].

Besides the propositions and corollaries mentioned above, in the sequel, some useful definitions and comments are presented, which play a crucial role in the eigenstructure analysis of nonlinear systems.

DEFINITION 2.1. Suppose there is an autonomous dynamical system with zero equilibrium point presented by the differential equation $\dot{\mathbf{x}}(t)=\mathbf{f}\left(\mathbf{x}(t)\right)$, in which $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^n$. Then, the region $D_A\subset\mathbb{R}^n$ is called 0-attracting if $\forall\mathbf{x}(\mathbf{0})\in\mathbb{R}^n$, all solution trajectories of the system move toward the origin, exponentially or spirally, when they are in the region $D_A\subset\mathbb{R}^n$. Subsequently, the region $D_R\subset\mathbb{R}^n$ is called 0-repelling, if $\forall\mathbf{x}(\mathbf{0})\in\mathbb{R}^n$, all solution trajectories of this system move away from the origin, exponentially or spirally, when the trajectories are in $D_R\subset\mathbb{R}^n$. Indeed, the 0-attracting (repelling) region is a region in the phase space in which the distance from the origin is monotonically decreasing (increasing) in some norm.

COMMENT 2.1. If a 0-attracting region includes the origin, it can be considered as the region of attraction for the origin equilibrium. Indeed, the origin could be conservatively considered as a locally asymptotically stable equilibrium point.

COMMENT 2.2. For a nonlinear system of order n, transformable to a PL form with SI NEVectors, if the geometric multiplicity of every multiple NEValue is equal to its corresponding algebraic multiplicity, the 0-attracting and 0-repelling regions could be respectively obtained as:

$$D_A=\left\{\mathbf{x}\in\mathbb{R}^n \,\middle|\, \mathrm{Re}\left\{\lambda_i(\mathbf{x})\right\}<0\,,\, i=1,2,\cdots,n\right\} \text{ and}$$

$$D_R=\left\{\mathbf{x}\in\mathbb{R}^n \,\middle|\, \mathrm{Re}\left\{\lambda_i(\mathbf{x})\right\}>0\,,\, i=1,2,\cdots,n\right\}.$$

As a final remark, we should notice that all the results mentioned above are unquestionably applicable if there exists a particular class of nonlinear systems having SI NEVectors. This point is indeed more highlighted by introducing the pseudo linearizable form and the specific conditions provided by the following remark.

REMARK 2.1. A nonlinear system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$, with $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^{n\times n}$, is defined to be in pseudo linearizable form if it can be expressed as one of the following forms.

$$1:\begin{cases}\dot{x}_1=g_1(\mathbf{x})x_1\\ \dot{x}_2=g_2(\mathbf{x})x_2\\ \quad\vdots\\ \dot{x}_n=g_n(\mathbf{x})x_n\end{cases};\quad g_i:\mathbb{R}^n\to\mathbb{R}\,,\ i=1,2,\ldots,n.$$

$$2:\begin{cases}\dot{x}_1=g_1(\mathbf{x})x_1\\ \dot{x}_2=g_2(\mathbf{x})x_2\\ \quad\vdots\\ \dot{x}_j=g_j(\mathbf{x})x_j-\omega_j x_{j+1}\\ \dot{x}_{j+1}=\omega_j x_j+g_j(\mathbf{x})x_{j+1}\\ \quad\vdots\\ \dot{x}_n=g_n(\mathbf{x})x_n\end{cases};\quad \begin{matrix}g_i:\mathbb{R}^n\to\mathbb{R}\,,\, i=1,2,\ldots,n\,,\\ \omega_j\in\mathbb{R}\,,\, j\in\{1,2,\ldots,n\}\end{matrix}$$

Note that the system represented in case 1 certainly leads to real NEValues, while case 2 leads to both real and complex NEValues. For each pair of (x_j,x_{j+1}) there are a pair of complex NEValues. It is evident that in case 2, there is no constraint on the number of these pairs. Indeed, the main application of this remark is in the synthesis of some nonlinear systems with a desired qualitative behavior.

Remind that the possibility of being represented in pseudo linearizable form is only the sufficient condition for having SI NEVectors. Besides, having the pseudo linearizable form guarantees that for every multiple NEValue, its geometric multiplicity is equal to its algebraic multiplicity. Indeed, if a nonlinear system is in pseudo linearizable form, it has no generalized NEVectors.

So far, the sufficient ability for global qualitative analysis of a special class of nonlinear autonomous systems has been provided in the paper. Though the condition of SI NEVectors is highly restrictive, the main benefit of this Eigen-structure based analysis tool could be in the generating or synthesizes some systems with desired qualitative behavior. Now, in the next section, this elegant ability is applied to generate chaotic systems systematically.

3- Main problem; Chaos generation

It is well known that the occurrence of chaotic behavior is related to the interplay between local instability and global boundedness. The local instability is responsible for the exponential divergence of nearby trajectories, while the global boundedness folds trajectories within the finite volume of the systems phase space. In addition, these trajectories never settle down to a periodic regime. The combination of these two mechanisms results in the high sensitivity of the system trajectories to the initial conditions.

This description may also include the torus (quasi-periodic) dynamics; however, it can easily be distinguished from a chaotic trajectory by considering its frequency spectrum or its dimension. Chaotic trajectories have a continuous frequency spectrum and non-integer dimension, while the periodic and quasi-periodic trajectories have a discrete Fourier spectrum and integer dimension. In this paper, as a golden key to generate conservative chaos, we focus on this fundamental qualitative characteristic of chaotic behavior; locally unstable and globally bounded.

3-1- Golden qualitative feature of a chaotic system: Stretching and folding

As is well known, continual stretching and folding is also one of the fundamental properties of chaotic dynamics. To be more precise, local instability is due to this stretching while the folding is responsible for global boundedness. Knowing the fact that synthesis of a hysteresis pattern by smooth nonlinear functions, at least is not so facile, this subsection is concentrated on synthesizing some nonlinear functions $\lambda_i(\mathbf{x}), i=1,2,3$ to generate the continual stretching and folding property in the system dynamics.

Again by the help of pseudo linearizable form, it is possible to produce such a behavior by a proper selection of NEValues. The following NEValues are designed to satisfy this qualitative behavior. This approach may lead to different choices of NEValues which satisfy the desired qualitative behavior; one

Table 1. Qualitative analysis of system (3.3)

	Region	Sign of $\mathrm{Re}\{\lambda(\mathbf{x})\}$	Qualitative Behavior	
Region 1	$\{\mathbf{x}\in\mathbb{R}^3 \mid ax_1^2+bx_2^2>r^2 \ \ \& \ \ x_3^3>h^2\}$	$\mathrm{Re}\{\lambda_{1,2}(\mathbf{x})\}>0$	$\frac{d}{dt}(\lvert x_1^2+x_2^2\rvert)>0$	Spiral
		$\mathrm{Re}\{\lambda_3(\mathbf{x})\}<0$	$\frac{d}{dt}(\lvert x_3\rvert)<0$	Exponential
Region 2	$\{\mathbf{x}\in\mathbb{R}^3 \mid ax_1^2+bx_2^2>r^2 \ \ \& \ \ x_3^3<h^2\}$	$\mathrm{Re}\{\lambda_{1,2}(\mathbf{x})\}<0$	$\frac{d}{dt}(\lvert x_1^2+x_2^2\rvert)<0$	Spiral
		$\mathrm{Re}\{\lambda_3(\mathbf{x})\}<0$	$\frac{d}{dt}(\lvert x_3\rvert)<0$	Exponential
Region 3	$\{\mathbf{x}\in\mathbb{R}^3 \mid ax_1^2+bx_2^2<r^2 \ \ \& \ \ x_3^3<h^2\}$	$\mathrm{Re}\{\lambda_{1,2}(\mathbf{x})\}<0$	$\frac{d}{dt}(\lvert x_1^2+x_2^2\rvert)<0$	Spiral
		$\mathrm{Re}\{\lambda_3(\mathbf{x})\}>0$	$\frac{d}{dt}(\lvert x_3\rvert)>0$	Exponential
Region 4	$\{\mathbf{x}\in\mathbb{R}^3 \mid ax_1^2+bx_2^2<r^2 \ \ \& \ \ x_3^3>h^2\}$	$\mathrm{Re}\{\lambda_{1,2}(\mathbf{x})\}>0$	$\frac{d}{dt}(\lvert x_1^2+x_2^2\rvert)>0$	Spiral
		$\mathrm{Re}\{\lambda_3(\mathbf{x})\}>0$	$\frac{d}{dt}(\lvert x_3\rvert)>0$	Exponential

of these choices is the following:

$$\begin{bmatrix}\lambda_{1,2}(\mathbf{x})=(x_3^2-h^2)\pm j\,\omega\\ \lambda_3(\mathbf{x})=r^2-ax_1^2-bx_2^2\end{bmatrix} \tag{3.2}$$

in which, r , b , a , ω and h are positive arbitrary real numbers. By applying remark 2.1, the following pseudo linearizable system can be obtained by these NEValues:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}(x_3^2-h^2)x_1-\omega x_2\\ \omega x_1+(x_3^2-h^2)x_2\\ (r^2-ax_1^2-bx_2^2)x_3\end{bmatrix} \tag{3.3}$$

The NEValuses of (3.2), as the qualitative behavior indicator of the system, dictate the desired continual stretching and folding on the system dynamics. It is just tried to get the system trajectories trapped in a cyclic manner. Indeed, this arbitrary pattern is selected to make the stability of spiral behavior in the x_1x_2 plane to be locked to the exponential behavior along the x_3 axis. The spiral behavior, which represents the complex NEValues, is the necessary component of a chaotic system. Selected NEValues are such that the system trajectories stretch along the x_3 axis within $-h\le x_3\le h$ and then fold when they reach the boundary of this region. Regarding non-uniqueness of this NEValues, (x_3^2-h) in $\lambda_{1,2}$ could be exchanged with $|x_3-h|$ or every other function of x_3 which behave similarly in the sense of sign change pattern. Similarly, in the λ_3 , $(r^2-ax_1^2-bx_2^2)$ could be substituted by $|r-ax_1-bx_2||r-ax_1-bx_2|$ or every other function of x_1 and x_2 with the same qualitative behavior. This behavior is to change sign across one closed region in the x_1x_2 plane. However, for the next of the paper, the system (3.3) is considered.

To be more precise, the NEVectors of the system (3.3), which are surely SI, are obtained as: $v_1=[1 \quad -j \quad 0]^T$, $v_2=[j \quad 1 \quad 0]^T$ and $v_3=[0 \quad 0 \quad 1]^T$. Based on the real parts of NEValuse, the state space of this system can be partitioned into four regions. The system's qualitative behavior in each region is given in Table 1. The validity of this type of analysis is based on the results of section 2 which is entirely given in [3]. The graphical illustration of the qualitative behavior of the nonlinear system (3.3) is also presented in Fig 3. Based on comment 2.2, the regions 2 and 4 of this nonlinear system are 0-attracting and 0-repelling, respectively. For chaos generation, the existence of these two regions both is necessary to ensure the stretching and folding

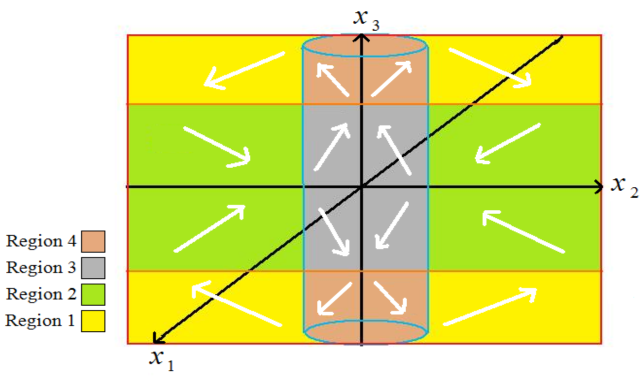

Fig. 3. Graphical illustration of the qualitative behavior of the system (3.3)

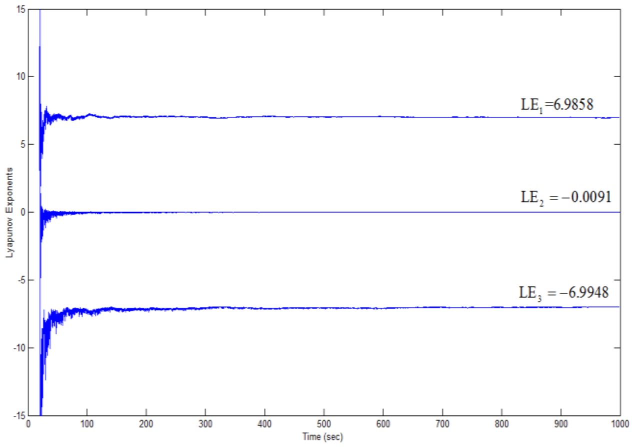

Fig. 4. Cyclic evolution of solution trajectories of the system (3.3)

of system trajectories. However, the sufficient condition is indeed the proper arrangement of these regions so that the cyclic and bounded qualities of system trajectories could be guaranteed. This condition is assured in the system (3.3), by the assistance of the regions 1 and 3, which complete the circle of system trajectories evolution. From these results, it can be deduced that the evolution of the system trajectories is based on the pattern illustrated in Fig. 4.

By this arrangement, it can be noticeably realized that the stretching of trajectories occur along the x_3 axis direction followed by the folding action and this pattern occurs repeatedly.

Qualitative analysis results of the system trajectories summarized in table 1 and Figs. 3-4, simply approve the system's almost global boundedness, which is thoroughly investigated in the next section. It worth mentioning that the proposed approach is primarily qualitative, so, without any quantitative manipulation, it is merely expected to generate a continual stretching and folding behavior. Consequently, it is expected that the synthesized system with differential equation (3.3) may have chaotic dynamics no matter what the system parameters' values are. For more elaboration, in the next section, through numerical simulation studies, the chaotic behavior of this system is approved and some interesting features of this chaotic system are revealed as well.

4- Analysis of numerical simulations

This section is devoted to showing the validity of the proposed qualitative approach for chaos generation through numerical simulations. Because our approach is essentially qualitative, the precise values of the parameters do not matter; hence, the following values are arbitrarily taken: $a = 5$, $b = 1$, $\omega = 50$, $h = 5$ and $r = 10$. By this assignment, the simulated 3D phase space of the system is given in Fig. 5. Since the $x_3 = 0$ is the invariant manifold of the system (stable manifold of the origin), system trajectories cannot cross it.

From these Figs, one can observe the chaos-like behavior of the system in some sense. The dynamic Lyapunov exponents of the system trajectories shown in Fig. 7 indicate the chaotic regime. The calculation of Lyapunov exponents is based on the well-known method proposed in [8].

The almost equality of absolute values of the positive and negative Lyapunov exponents means that the expansion power of trajectories is nearly equal to their contraction power. Indeed, this could be a clue of a conservative chaotic behavior, which can be more highlighted by calculating the dimension of the time series generated by the system.

The dimension of a strange attractor is a measure of its geometric scaling properties or its complexity and has been considered as the most fundamental property of an attractor. Numerous methods have been proposed for characterizing the fractional dimension of the strange attractors produced by chaotic systems. These methods fall into two broad categories; those derived from the topology, and those derived from the dynamics. Perhaps the most common of the former metrics is the correlation dimension popularized by [9] while the most common of the latter is the Lyapunov dimension proposed by [10]. Kaplan and Yorke introduced a quantity defined in terms of the Lyapunov exponents, LE_i, $i = 1,2,\ldots,n$, where the subscript labeling of the LE_i is chosen so that $LE_1 > LE_2 > \cdots > LE_n$. The quantity introduced by Kaplan and Yorke is commonly called the 'Lyapunov dimension' and is given by:

$$D_L = k + \frac{1}{|LE_{k+1}|}\sum_{j=1}^{k} LE_j$$

where k is the largest integer for which $LE_1 + LE_2 + \cdots LE_k > 0$.

Based on this relation, the Lyapunov dimension of the chaotic system trajectories, shown in Fig.5, is $D_L = 2.9974$. This almost integer Lyapunov dimension verifies the conservative nature of the system behavior. The time series plot of $x_1(t)$ is also illustrated in Fig. 8. As expected for a conservative chaotic dynamics, there is no transient in the system response. Indeed, while the system is in the chaotic regime for all times, its response will never settle down to an attractor. To be more precise about the chaotic behavior of

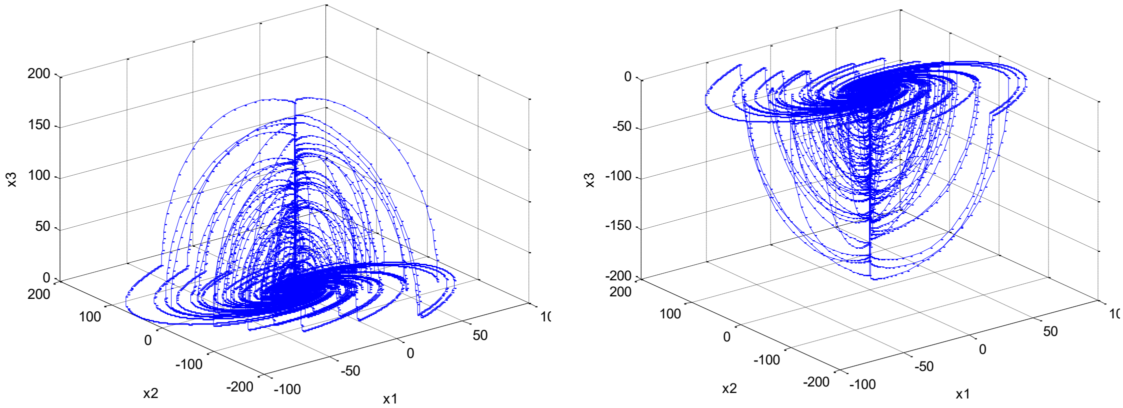

Fig. 6. 3D phase plane plot of the system (3.3); Left: , Right:

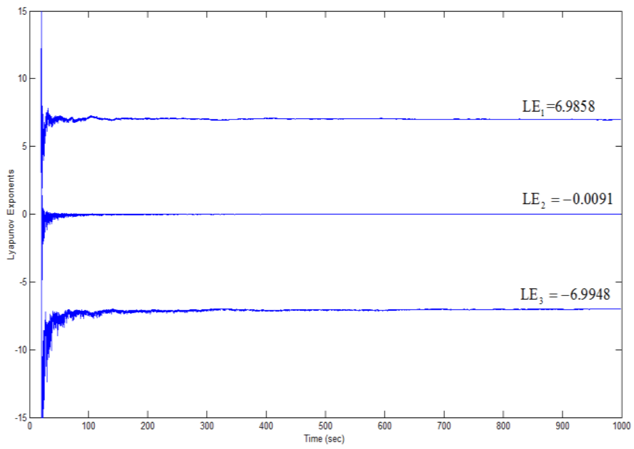

Fig. 7. Dynamic Lyapunov exponent of system trajectories depicted in fig. 6.

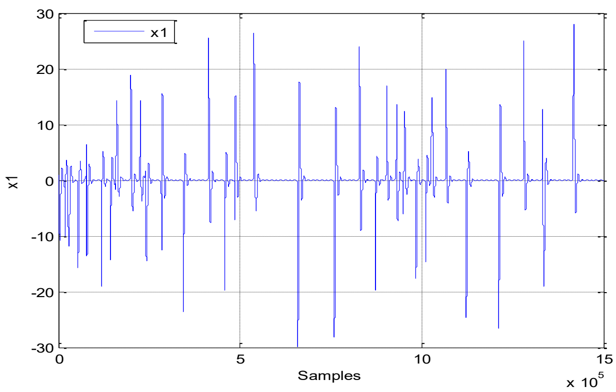

Fig. 8. Time series plot of $x_1(t)$; for system trajectories depicted in fig. 6.

this derived system, its frequency spectrum is illustrated in Fig. 9.

As was expected, the continuous power spectrum verifies the qualitative analysis results about the chaotic behavior of the system. It is worth mentioning that this system may have even a larger Lyapunov exponent for some other parameters' values. In addition, the shape of the system's 3D trajectory can be arbitrarily changed by an arbitrarily tuning of the system parameters. For example, to make the global 3D trajectory more stretched along the x_3 axis, the parameter h should be larger and the similar reshaping along the x_1x_2 plane should be made too, via parameters r, a and b.

The other elegant feature of this chaotic system is that the chaotic behavior is obtained for almost all parameters' values and for all initial conditions except those located on the manifold $\{\mathbf{x}\in\mathbb{R}^3 \mid x_3=0\}$, the stable manifold of the origin and those located on the manifold $\{\mathbf{x}\in\mathbb{R}^3 \mid x_1=0 \quad \& \quad x_2=0\}$, which is indeed the unstable manifold of origin.

The major drawback of a conservative chaotic flow may be the possibility of its unboundedness due to the lack of an attractor. Indeed, this lack of attractors is both a simplification and a complication. Since there are no attractors, there is no worry about transients; that is, it is not needed to let the trajectory runs for some time so that it settles down onto the appropriate attractor. This trait usually simplifies the process of finding the appropriate solution for the trajectories. On the other hand, the lack of attractors means that trajectories starting with different initial conditions may behave quite differently as time goes on; there is no universal attractor onto which they settle down. However, for the proposed system, except those trajectories with initial conditions located precisely on the stable and unstable manifolds, all other trajectories will have the almost same qualitative behavior. Specifically, with the selected parameter's values, the system behavior is similar to the almost the same characteristics. That is why the initial conditions are not mentioned in the simulation results. In the sequel, the global boundedness of all trajectories for all other initial conditions is assured.

Based on the qualitative analysis of the system (3.3) depicted in Table 1, Fig. 3 and Fig.4, the system trajectories will be unbounded if either the initial conditions are located exactly on the unstable manifold of origin, i.e. $\{\mathbf{x}\in\mathbb{R}^3 \mid x_1=0 \quad \& \quad x_2=0\}$, or there exists some finite time $t^{\#}$ such that the system trajectory will intersect the unstable

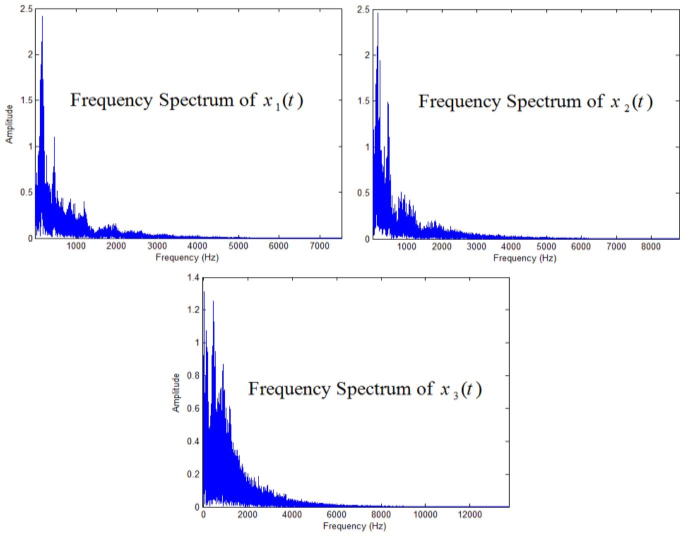

Fig. 9. The frequency spectrum of system trajectories depicted in fig. 6.

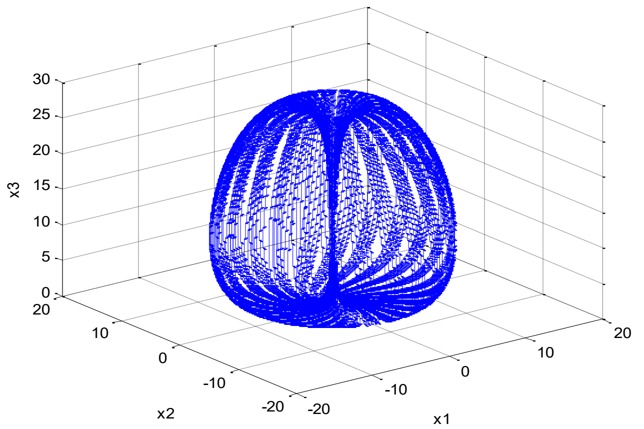

Fig. 10. Torus generated by the system (3.3) with: $a=5$, $b=5$, $\omega=50$, $h=10$ **and**

manifold, the x_3 axis, at that time. The first case can be simply prevented by excluding this manifold from permitted initial conditions while the second one will never happen as will be shown below. The qualitative behavior of the nonlinear system in the x_1x_2 - plane is a spiral with an amplitude of $K\exp\left((x_3^2-h^2)t\right); k\in\mathbb{R}$ and with an angular frequency of ω while the parameters h and ω are arbitrarily selected. Even if the exponent term, i.e. $(x_3^2-h^2)t$, is negative for all $t\geq 0$, the system trajectory approaches the x_3 axis as t goes to infinity. However, in our case, the exponent term is negative only for some finite time $t^{\otimes}$ at which the system trajectory intersects the planes $x_3=\pm h$ and eventually leaves the region 3. Thus, as depicted in the graphical illustration of the system behavior in Fig. 3, the system trajectory converges to the x_3 axis in region 3 and then, at some time say $t^{\otimes}$, it leaves this region to enter region 4 and finally it diverges from the x_3 axis in region 4. In short, even if the initial conditions are located very near to - but not precisely on - the unstable manifold, the system trajectory does never cross the x_3 axis.

The last point, which worth mentioning is that based on the proposed qualitative analysis, the system (3.3) can also generate some torus-like chaotic dynamics. This phenomenon is observed when the system parameters are chosen symmetrically so that the resulting system trajectory has a symmetric shape. For example, if the parameters a and b are selected identically the same, the system may produce something like torus dynamics. Similar behavior may be obtained by equal selecting of the parameters h and r. Indeed, as stated in section 3, this behavior is again expected, due to the fact the presented approach for generating chaos is mainly based on the continual stretching and folding without settling down to a periodic orbit, which is also the essential characteristic of the torus-like behavior. On the other hand, it is known that an autonomous torus could be destroyed and transformed into chaos by applying a perturbation or by destroying its symmetry, [11].

Indeed, it is only needed to break down the aforementioned symmetries to have a chaotic behavior. That is one reason for the specified parameters' values selected for the first simulations. To highlight this point better, we take the following symmetric parameters' values: $a=5$, $b=5$, $\omega=50$, $h=10$ and $r=10$. The results given in Fig 10, Fig. 11 and Fig. 12 facilely illustrate the generated torus-like chaotic behavior. The shape of the 3D trajectory plot in Fig. 10 and the frequency spectrum of the system trajectories depicted in Fig. 12 indicate that this type of behavior is very similar to a torus dynamics.

Based on the KAM theorem [12], this chaotic behavior indicated by the very small positive Lyapunov exponents for symmetric parameters' values, though presented in principle,

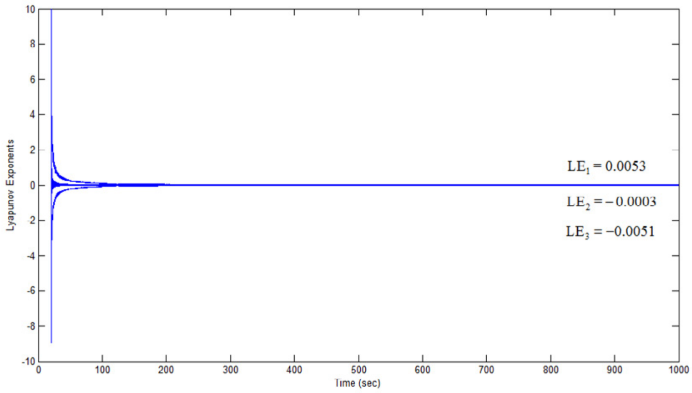

Fig. 11. Dynamic Lyapunov exponent of system trajectories depicted in fig. 10

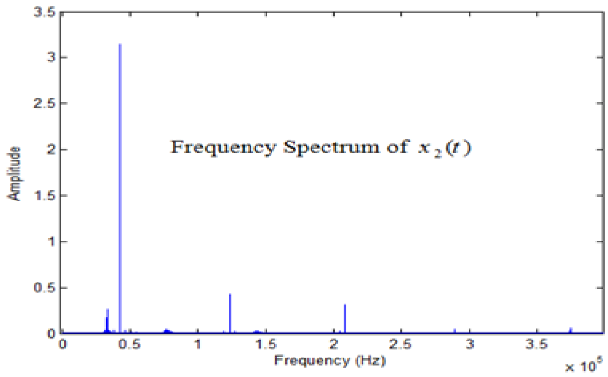

Fig. 11. Dynamic Lyapunov exponent of system trajectories depicted in fig. 10

may not be noticeable at a practical level.

As stated earlier, the destruction of symmetry of parameters' values could be considered as a perturbation applied to the torus-like chaotic dynamic with symmetric parameters and makes the system non-integrable and then chaos emerges more obviously. This chaos generation scheme, to some extent, corresponds to the well-known KAM theorem ([13]).

All in all, this paper aimed to employ the previously eigen-structure analysis method to generate some nonlinear system with really complex behavior. The lowest level of complexity which could be obtained is some torus-like behavior, and the highest one is a conservative chaotic behavior with a very high positive Lyapunov exponent. Indeed, it was tried to highlight the ability to force some nonlinear systems to behave in desired quality through well-known chaos generation problem. However, there are many other challenging issues in nonlinear control theory, which might be tackled by this privileged ability. As an example, another dissipative chaotic system generated by this method and its dynamical analysis have been proposed in [14] as well as its eigen-structure based synchronization problem.

5- Conclusion and remarks

Qualitative analysis of nonlinear autonomous dynamic systems for chaos generation was the basic idea of this paper. In this regard, for a particular class of nonlinear systems named as pseudo linearizable form, sufficient tools for eigenstructure analysis were fully developed with the help of some new concepts of NEValuse and NEVectors. The approach proposed in the paper, under some sufficient conditions, guarantees to reach the globally valid results. Since the continual stretching and folding could be considered as the essential qualitative characteristic of chaotic behavior, the final and primary step of this paper was dedicated to synthesizing this chaotic qualitative characteristic by the help of the aforementioned qualitative analysis tools. This approach led to a nonlinear autonomous system with apparently chaotic behavior. The systematic generation approach proposed in this paper possesses an elegant feature that there is no trial and error process to find the parameters' values of the derived chaotic system. Indeed, except for some symmetric arrangement of parameters' values leading to some torus-like chaotic dynamics, almost for all parameters and all initial conditions, the derived system exhibits different levels of chaotic behavior. This parameter independency of this chaos generation scheme is owed to the qualitative attribute of the proposed approach. Although the resulting chaotic system was conservative, its global boundedness along with its property of not settling down to a regular attractor can be guaranteed. Indeed, the conservative feature of the obtained chaotic system is because of the fact that merely the continual stretching and folding feature is tried to be synthesized; without forcing the trajectories to settle down to an attractor. In fact, applying the proposed global qualitative analysis tool helps us generate different nonlinear systems with desired qualitative behaviors.

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