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The Exact Solution of Min-Time Optimal Control Problem in Constrained LTI Systems: A State Transition Matrix Approach

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ABSTRACT: In this paper, the min-time optimal control problem is mainly investigated in the linear time invariant (LTI) continuous-time control system with a constrained input. A high order dynamical LTI system is firstly considered for this purpose. Then the Pontryagin principle and some necessary optimality conditions have been simultaneously used to solve the optimal control problem. These optimality conditions would usually lead to some complicated equations while some integral terms may be presented. Then a systematic procedure based on state transition matrix will be addressed to overcome and simplify the mentioned complexities. Therefore the state transition matrix would be used to determine the exact solution of the min-time control problem in a typical LTI system. The min-time problem would be converted to some algebraic nonlinear equations by using of the state transition matrix. These algebraic equations are depended on some definite parameters. Hence the required design parameters as well as switching times and the possible minimum time would be analytically determined in the minimum-time optimal control problem. Thus the min-time control signal would be explicitly determined by computing of the switching times and also some other constants. The proposed control scheme is applied in some typical dynamical examples to show the effectiveness of the suggested control method.

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1- Introduction

The optimal control problems have attracted the attention of many researchers and engineers in the last decades [1, 2]. In the optimal control theory, the input signal would be selected such that a predefined cost function is minimized while some constraints are presented. Such a cost function maybe a physical quantity like energy, fuel, final time and the others. Many optimal control problems could be mathematically formulated by deriving of some necessary and sufficient optimality conditions. But there is not a systematic way to find the exact solution of the optimal control problems either analytically or numerically. Hence the approximated solutions have been interested in the control applications [3, 4].

Sometime the optimal control problem may have not a unique solution. In other words, they maybe have either many solutions or no solution exists. There are some examples which the optimal control law does not exist. For an example, the min-time problem has not an optimum solution in unconstrained LTI system. There would be a unique solution to the optimal control problem in the unconstrained LTI systems. But determination of the exact optimum solution via analytical way maybe have some mathematical difficulties due to emerge of two-point boundary value problem [1, 5]. Beside of such a complexity, sometimes it is preferred to look for a suboptimal or a near-to-optimal control rather than the exact solution [6, 7]. The numerical method as well as the wavelet functions may be used to find an approximated solution [8].

The min-time problems have been increasingly interested in some applications like aerospace, teleportation and the others [9, 10]. In such problems, the input signal is designed such that the goal would be satisfied in a minimum feasible time [11-13]. The well-known bang-bang control usually leads to a min-time control [14]. The solution of the min-time problem may have some mathematical complexities in comparing with the other optimal control problem as well as minimum fuel or energy. The well-known phase plane drawing methods would be a graphical approach to find the min-time control law in a simple second order dynamical system. Although the phase plane method is very helpful but it could not be extended to the high order systems [1].

There are some control applications as well as the guidance system which may be treated as a typical LTI system [15, 16]. Hence the exact solution computation would be useful in such min-time optimal problem. Lately the min-time problem has been formulated in the continuous-time LTI systems with real poles. Then the optimal control signal may be numerically found in the LTI system [17]. This study motivates the author to develop an analytical and systematic framework to the exact solution of the min-time control problem. Therefore a systematic procedure based on the state transition matrix is addressed in this paper. For achieving

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this goal, firstly, the min-time problem is mathematically formulated in the constrained LTI system. Then it is converted to an algebraic nonlinear equation which depends on some parameters like the switching times and also some other constants. The switching times and also the minimum time would be calculated by the solution of the proposed equation analytically or numerically. Therefore the optimal control signal can be explicitly determined in a typical LTI systems.

The rest of this paper is organized as follows: the min-time control problem is firstly formulated in the next Section. Then the main results are presented in Section 3. Three numerical examples are investigated in Section 4. Some concluding remarks are places is the last Section.

2- Problem Statement

Consider the following constrained LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t), |u(t)| \le 1 \tag{1}$$

where $x(t) \in R^n$ denotes the system state and $u(t) \in R$ is a control input. The constrained LTI system (1) is considered for formulation of the min-time problem. Then it is desired to find a piecewise-continuous control signal $u(t)$, $t \in [t_0, t_f]$ such that the system states can be driven from an initial state $x(t_0) = x_0$ to a final state $x(t_f) = x_f$ in a minimum possible time. It is trivial that without loss of generality the initial time t_0 may be set as zero. In order to formulate the min-time problem, the following assumption is considered:

Assumption: The pair (A, B) is controllable in the dynamical system (1).

By considering of the optimality conditions (for more detail see [1]), it is shown that there exists at most $n-1$ switching times in the input signal [17]. The Pontryagin principle implies $|u(t)| = 1$ in order to obtain a min-time control law [18]. Then the switching times may be defined as t_k, $k = 1,2,\dots,n-1$. Let define a pulse function $P_{i,j}(t)$ as the following:

$$P_{i,j}(t) = \begin{cases} 1 & t_i \le t < t_j \\ 0 & t < t_i \ \ and \ \ t \ge t_j \end{cases} \tag{2}$$

Then the control signal $u(t)$ could be represented as follows:

$$u(t) = \sum_{k=0}^{n-1} \alpha_k P_{k,k+1}(t) \tag{3}$$

where $\alpha_k = \pm 1$, $k = 0, 1, 2, \dots, n-1$. It is desired to drive the system states from an initial point x_0 to the final state x_f in a minimum possible time. It is easy to check that there exists at most 2^n signal profiles for the control input $u(t)$. The min-time problem would be solved in the constrained LTI system (1) if the switching times t_k, $k = 1, 2, \dots, n-1$ and some constants $\alpha_k = \pm 1$, $k = 0, 1, 2, \dots, n-1$ are suitably found. Next, a systematic procedure is suggested to determine the optimal signal $u(t)$ in the min-time problem.

3- Main Result

The min-time control problem of the constrained LTI system (1) would be solved in this Section by using of the state transition matrix. For this purpose, the min-time problem is converted to an algebraic nonlinear equation which depends on the switching times t_k and also some constants α_k. Since already $|\alpha_k| = 1$ then there exists at most 2^n signal profiles for the control input $u(t)$. The algebraic nonlinear equation could be solved explicitly or numerically by assuming α_k to be some constant values.

Theorem 1: Suppose that the matrix A is invertible and the constants $\alpha_k = \pm 1$, $k = 0,1,2,\dots,n-1$ are given. If there exists a solution to the following algebraic equation:

$$\sum_{k=0}^{n-1} \Phi(t_k') \beta_k = x_f' \tag{4}$$

where $\Phi(t)$ denotes the state transition matrix and the parameters t_k', β_k and x_f' are defined as:

$$\begin{aligned} t_k' &= t_n - t_k, k = 0,1,2,\dots,n-1 \\ \beta_0 &= x_0 + \alpha_0 A^{-1} B \\ \beta_k &= (\alpha_k - \alpha_{k-1}) A^{-1} B, k = 1,2,\dots,n-1 \\ x_f' &= x_f + \alpha_{n-1} A^{-1} B \end{aligned}$$

then the minimum feasible time $t_n = t_f$ and the switching times t_k, $k = 1,2,\dots,n-1$ would be determined. Therefore $u(t) = \sum_{k=0}^{n-1} \alpha_k P_{k,k+1}(t)$ would be a min-time control which moves the dynamical system (1) from the initial condition $x(t_0) = x_0$ to the final state $x(t_f) = x_f$.

Proof: The system equation which described by the differential equation (1) is written as the following [19]:

$$\frac{d}{dt}\left(e^{-At} x(t)\right) = e^{-At} Bu(t) \tag{5}$$

By integrating both sides of Eq. (5) from $t = t_k$ to $t = t_{k+1}$ with considering $u(t) = \alpha_k$, $t_k \le t < t_{k+1}$, we have:

$$e^{-At} x(t) \Big|_{t_k}^{t_{k+1}} = -\alpha_k e^{-At} A^{-1} B \Big|_{t_k}^{t_{k+1}} \tag{6}$$

Then Eq. (6) is simplified as follows:

$$e^{-At_{k+1}} x(t_{k+1}) - e^{-At_k} x(t_k) = \alpha_k e^{-At_k} A^{-1} B - \alpha_k e^{-At_{k+1}} A^{-1} B \tag{7}$$

Then $x(t_{k+1})$ in term of $x(t_k)$ could be obtained as:

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)} x(t_k) + \alpha_k e^{A(t_{k+1}-t_k)} A^{-1} B - \alpha_k A^{-1} B, k = 0,1,2,\dots,n-1 \tag{8}$$

It is seen that the term $x(t_1)$, corresponding to $k = 0$, may be written as:

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) + \alpha_0 e^{A(t_1-t_0)}A^{-1}B - \alpha_0\ A^{-1}B \quad (9)$$

It can be simplified as follows:

$$x(t_1) = e^{A(t_1-t_0)}\left(x(t_0) + \alpha_0 A^{-1}B\right) - \alpha_0\ A^{-1}Ba \quad (10)$$

Similarly the vector $x(t_2)$ is obtained as:

$$x(t_2) = e^{A(t_2-t_1)}x(t_1) + \alpha_1 e^{A(t_2-t_1)}A^{-1}B - \alpha_1\ A^{-1}B \quad (11)$$

It may be rewritten as follows:

$$x(t_2) = e^{A(t_2-t_0)}\left(x(t_0) + \alpha_0 A^{-1}B\right) + (\alpha_1 - \alpha_0)e^{A(t_2-t_1)}A^{-1}B - \alpha_1\ A^{-1}B \quad (12)$$

The vector $x(t_3)$ can also be obtained as:

$$x(t_3) = e^{A(t_3-t_2)}x(t_2) + \alpha_2 e^{A(t_3-t_2)}A^{-1}B - \alpha_2\ A^{-1}B \quad (13)$$

Then

$$x(t_3) = e^{A(t_3-t_0)}\left(x(t_0) + \alpha_0 A^{-1}B\right) + (\alpha_1 - \alpha_0)e^{A(t_3-t_1)} A^{-1}B + (\alpha_2 - \alpha_1)e^{A(t_3-t_2)}\ A^{-1}B - \alpha_2\ A^{-1}B \quad (14)$$

Finally the term $x(t_n)$ will be formulated as:

$$x(t_n) = e^{A(t_n-t_0)}\left(x(t_0) + \alpha_0 A^{-1}B\right) + (\alpha_1 - \alpha_0)e^{A(t_n-t_1)}A^{-1}B + (\alpha_2 - \alpha_1)e^{A(t_n-t_2)} A^{-1}B + (\alpha_3 - \alpha_2)e^{A(t_n-t_3)}\ A^{-1}B + \ldots + (\alpha_{n-1} - \alpha_{n-2})e^{A(t_n-t_{n-1})}\ A^{-1}B - \alpha_{n-1}\ A^{-1}B \quad (15)$$

By considering $x(t_n) = x_f$, Eq. (15) would be rewritten as follows:

$$x_f + \alpha_{n-1}\ A^{-1}B = e^{A(t_n-t_0)}\left(x_0 + \alpha_0 A^{-1}B\right) + (\alpha_1 - \alpha_0)e^{A(t_n-t_1)}A^{-1}B + (\alpha_2 - \alpha_1)e^{A(t_n-t_2)}\ A^{-1}B + (\alpha_3 - \alpha_2)e^{A(t_n-t_3)}\ A^{-1}B + \ldots + (\alpha_{n-1} - \alpha_{n-2})e^{A(t_n-t_{n-1})}\ A^{-1}B \quad (16)$$

Let define some variables as follows:

$$t_k' = t_n - t_k, k = 0,1,2,\ldots,n-1$$
$$\beta_0 = x_0 + \alpha_0 A^{-1}B$$
$$\beta_k = (\alpha_k - \alpha_{k-1})A^{-1}B, k = 1,2,\ldots,n-1$$
$$x_f' = x_f + \alpha_{n-1}\ A^{-1}B$$

Thus Eq. (16) would be written as:

$$x_f' = e^{At_0'}\beta_0 + e^{At_1'}\beta_1 + e^{At_2'}\ \beta_2 + e^{At_3'}\ \beta_3 + \ldots + e^{At_{n-1}'}\beta_{n-1} \quad (17)$$

The state transition matrix $\Phi(t)$ is found as the following [19]:

$$\Phi(t) = e^{At} = L^{-1}\left\{(sI_n - A)^{-1}\right\} \quad (18)$$

Hence the following equation would be concluded:

$$x_f' = \Phi(t_0')\beta_0 + \Phi(t_1')\beta_1 + \Phi(t_2')\ \beta_2 + \Phi(t_3')\beta_3 + \ldots + \Phi(t_{n-1}')\beta_{n-1} \quad (19)$$

Then Eq. (4) can be obtained. It completes the proof.

It is clear Eq. (19) would actually be a set of the n-nonlinear equations and n-unknown variables. Therefore t_k', $k = 0,1,2,\ldots,n-1$ can be computed explicitly or numerically. Hence the switching times t_k, $k = 1,2,\ldots,n$ and also the min-time $t_n = t_f$ would be obtained.

Algorithm 1: Min-time control solution with invertible A

Step 1: Choose a suitable combination of the constants $\alpha_k = \pm 1$, $k = 0,1,2,\ldots,n-1$.

Step 2: Construct the vectors x_f', t_k' and β_k, $k = 0,1,2,\ldots,n-1$.

Step 3: Compute state transition matrix $\Phi(t)$.

Step 4: Solve the algebraic equation $x_f' = \sum_{k=0}^{n-1}\Phi(t_k')\beta_k$.

Step 5: Check the switching times t_k, $k = 1,2,\ldots,n$ are correctly obtained else go to the step 1 and tries with another combination of the constants α_k, $k = 0,1,2,\ldots,n-1$.

Remark 1: There would be no switching time in a special case $\alpha_0 = \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1}$, and Eq. (19) may be simplified as follows:

$$x_f + \alpha_0\ A^{-1}B = \Phi(t_n)\left(x_0 + \alpha_0 A^{-1}B\right) \quad (20)$$

Then only the minimum time $t_n = t_f$ could be computed.

Remark 2: In the first order systems, the minimum time t_f may be calculated as follows:

$$t_f = A^{-1}\ln\left(\frac{Ax_0 + \alpha_0 B}{Ax_f + \alpha_0\ B}\right) \quad (21)$$

where $\alpha_0 = +1$ or $\alpha_0 = -1\alpha_0 = -1$.

In Theorem 1, a systematic method is suggested to the min-time control problem in the constrained LTI system. The results of Theorem 1 would be failed when the matrix A is not invertible. Hence Theorem 1 would be extended in the next subsequent.

Theorem 2: Suppose $\alpha_k = \pm 1$, $k = 0,1,2,\ldots,n-1$ are some known constants. The minimum feasible time $t_n = t_f$ and the switching times t_k, $k = 1,2,\ldots,n-1$ would be determined if there exists a solution to the following algebraic equation:

$$\sum_{k=0}^{n-1}\gamma_k\Omega(t_k') = \tilde{x}_f \quad (22)$$

where $\Omega(t) = \int_{t_0}^{t}\Phi(\tau)Bd\tau$, $\Phi(t)$ is the state transition matrix and the parameters t_k', γ_k and $\tilde{x}_f$ are defined as:

$$t_k^{'} = t_n - t_k, k = 0,1,2,\ldots,n-1$$
$$\gamma_0 = \alpha_0$$
$$\gamma_k = \alpha_k - \alpha_{k-1}, k = 1,2,\ldots,n-1$$
$$\tilde{x}_f = x_f - \Phi(t_f)x_0$$

Therefore the signal $u(t) = \sum_{k=0}^{n-1} \alpha_k P_{k,k+1}(t)$ would be a min-time optimal control.

Proof: Let integrate both side of Eq. (5) from $t = t_k$ to $t = t_{k+1}$ and consider $u(t) = \alpha_k$, $t_k \le t < t_{k+1}$. Then $x(t_{k+1})$ in term of $x(t_k)$ is obtained as follows:

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \alpha_k \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bd\tau, \quad k = 0,1,2,\ldots,n-1 \tag{23}$$

The term $x(t_1)$ may be written as:

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) + \alpha_0 \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bd\tau \tag{24}$$

It can be simplified as follows:

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) + \alpha_0 \int_{0}^{t_1-t_0} e^{A\tau}Bd\tau \tag{25}$$

Similarly the vector $x(t_2)$ is obtained as:

$$x(t_2) = e^{A(t_2-t_0)}x(t_0) + \alpha_0 \int_{0}^{t_1-t_0} e^{A(t_2-t_1+\tau)}Bd\tau + \alpha_1 \int_{t_1}^{t_2} e^{A(t_2-\tau)}Bd\tau \tag{26}$$

It may be rewritten as follows:

$$x(t_2) = e^{A(t_2-t_0)}x(t_0) + \alpha_0 \int_{t_2-t_1}^{t_2-t_0} e^{A\tau}Bd\tau + \alpha_1 \int_{0}^{t_2-t_1} e^{A(t_2-\tau)}Bd\tau \tag{27}$$

The vector $x(t_3)$ can be also obtained as:

$$x(t_3) = e^{A(t_3-t_2)}x(t_2) + \alpha_2 \int_{t_2}^{t_3} e^{A(t_3-\tau)}Bd\tau \tag{28}$$

Then

$$x(t_3) = e^{A(t_3-t_0)}x(t_0) + \alpha_0 \int_{t_3-t_1}^{t_3-t_0} e^{A\tau}Bd\tau + \alpha_1 \int_{t_3-t_2}^{t_3-t_1} e^{A\tau}Bd\tau + \alpha_2 \int_{0}^{t_3-t_2} e^{A\tau}Bd\tau \tag{29}$$

Finally the term $x(t_n)$ will be formulated as:

$$x(t_n) = e^{A(t_n-t_0)}x(t_0) + \alpha_0 \int_{t_n-t_1}^{t_n-t_0} e^{A\tau}Bd\tau + \alpha_1 \int_{t_n-t_2}^{t_n-t_1} e^{A\tau}Bd\tau + \alpha_2 \int_{t_n-t_3}^{t_n-t_2} e^{A\tau}Bd\tau + \ldots + \alpha_{n-2} \int_{t_n-t_{n-1}}^{t_n-t_{n-2}} e^{A\tau}Bd\tau + \alpha_{n-1} \int_{0}^{t_n-t_{n-1}} e^{A\tau}Bd\tau \tag{30}$$

Eq. (30) would be rewritten as follows:

$$x_f = \Phi(t_f)x(t_0) + \alpha_0\Omega(t_0^{'}) + (\alpha_1 - \alpha_0)\Omega(t_1^{'}) + (\alpha_2 - \alpha_1)\ \Omega(t_2^{'}) + \ldots + (\alpha_{n-1} - \alpha_{n-2})\Omega(t_{n-1}^{'}) \tag{31}$$

Then Eq. (22) can be obtained. It completes the proof.

Algorithm 2: Min-time control solution in general form

Step 1: Choose a suitable combination of the constants $\alpha_k = \pm 1, k = 0,1,2,\ldots,n-1$.

Step 2: Construct the vectors $\tilde{x}_f$, t_k' and γ_k, $k = 0,1,2,\ldots,n-1$.

Step 3: Compute state transition matrix $\Phi(t)$ and vector $\Omega(t)$.

Step 4: Solve the algebraic equation $\sum_{k=0}^{n-1} \gamma_k\ \Omega(t_k') = \tilde{x}_f$.

Step 5: Check the switching times t_k, $k = 1,2,\ldots,n$ are correctly obtained else go to the step 1 and tries with another combination of the constants α_k, $k = 0,1,2,\ldots,n-1$.

Remark 3: There would be no switching time in a special case $\alpha_0 = \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1}$. Then Eq. (22) may be rewritten as follows:

$$\alpha_0\Omega(t_f) + \Phi(t_f)x_0 = x_f \tag{32}$$

Then the minimum time $t_n = t_f$ can be computed.

Remark 4: The vector $\Omega(t) = (e^{At} - e^{At_0})A^{-1}B$ would be found when the matrix A is invertible. Hence the results of Theorem 1 can be imagined as a special case of Theorem 2 with invertible A.

Next the proposed procedure is applied in some numerical examples.

4- Numerical Simulation

In this Section, the proposed procedure is used in some dynamical examples (i.e. the LTI systems with real poles, complex poles and imaginary poles) to find a min-time control signal.

Example 1. Consider the following continuous-time LTI system:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t) \tag{33}$$

The initial and final conditions of the dynamical system (33) are selected as follows:

$$y(0) = 1, \dot{y}(0) = 0, y(t_f) = 0, \dot{y}(t_f) = 0$$

It is desired to find a min-time optimal control $u(t)$. Then

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The poles of the LTI system (33) are some real values. Hence the algorithm 1 can be used in this example. The terms β_0, β_1 and x_f' are determined as follows:

$$\beta_0 = \frac{2-\alpha_0}{2}\begin{bmatrix}1\\0\end{bmatrix}, \beta_1 = -\frac{\alpha_1-\alpha_0}{2}\begin{bmatrix}1\\0\end{bmatrix}, x_f' = -\frac{1}{2}\alpha_1\begin{bmatrix}1\\0\end{bmatrix}$$

The state transition matrix $\Phi(t)$ may be computed as:

$$\Phi(t) = e^{At} = L^{-1}\left\{\begin{bmatrix} s & -1 \\ 2 & s+3\end{bmatrix}^{-1}\right\}$$

Then

$$\Phi(t) = L^{-1}\left\{\begin{bmatrix} \frac{2}{s+1}-\frac{1}{s+2} & \frac{1}{s+1}-\frac{1}{s+2} \\ \frac{-2}{s+1}+\frac{2}{s+2} & \frac{-1}{s+1}+\frac{2}{s+2}\end{bmatrix}\right\} =$$

$$\begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

The min-time control problem can be solved by constructing of the following equation:

$$x_f' = \Phi(t_0')\beta_0 + \Phi(t_1')\beta_1$$

where $t_0' = t_2 \geq 0$ and $t_1' = t_2 - t_1 \geq 0$. Then

$$\alpha_1\begin{bmatrix}1\\0\end{bmatrix} = (\alpha_0-2)\Phi(t_0')\begin{bmatrix}1\\0\end{bmatrix} + (\alpha_1-\alpha_0)\Phi(t_1')\begin{bmatrix}1\\0\end{bmatrix} \quad (34)$$

It can be written as follows:

$$\alpha_1\begin{bmatrix}1\\0\end{bmatrix} = (\alpha_0-2)\begin{bmatrix} 2e^{-t_0'}-e^{-2t_0'} \\ -2e^{-t_0'}+2e^{-2t_0'}\end{bmatrix} + (\alpha_1-\alpha_0)\begin{bmatrix} 2e^{-t_1'}-e^{-2t_1'} \\ -2e^{-t_1'}+2e^{-2t_1'}\end{bmatrix} \quad (35)$$

Eq. (35) would be simplified as:

$$\begin{cases} (\alpha_0-2)\left(2e^{-t_0'}-e^{-2t_0'}\right) + \\ (\alpha_1-\alpha_0)\left(2e^{-t_1'}-e^{-2t_1'}\right) = \alpha_1 \\ (\alpha_0-2)\left(-2e^{-t_0'}+2e^{-2t_0'}\right) + \\ (\alpha_1-\alpha_0)\left(-2e^{-t_1'}+2e^{-2t_1'}\right) = 0 \end{cases} \quad (36)$$

In this example, there are 4 different cases for the constants α_0 and α_1. The solution of Eq. (36) can be only computed with $\alpha_0 = -1$ and $\alpha_1 = 1$. In this case, Eq. (36) would be written as follows:

$$\begin{cases} -6e^{-t_0'} + 3e^{-2t_0'} + \\ 4e^{-t_1'} - 2e^{-2t_1'} = 1 \\ 6e^{-t_0'} - 6e^{-2t_0'} - \\ 4e^{-t_1'} + 4e^{-2t_1'} = 0 \end{cases} \quad (37)$$

It can be rewritten as:

$$\begin{cases} 2e^{-t_1'} - 3e^{-t_0'} = 1 \\ 2e^{-2t_1'} - 3e^{-2t_0'} = 1 \end{cases} \quad (38)$$

Thus it is not hard to show that $t_1' = Ln(\frac{\sqrt{3}+1}{2}) \approx 0.3119$ and $t_0' = Ln\left(2\sqrt{3}+3\right) \approx 1.8663$ are the unique solutions. In the other cases, Eq. (36) has not a real solution. It can be simply checked as follows:

In the second case $\alpha_0 = \alpha_1 = 1$, Eq. (36) would be simplified as:

$$\begin{cases} 2e^{-t_0'} - e^{-2t_0'} + 1 = 0 \\ -2e^{-t_0'} + 2e^{-2t_0'} = 0 \end{cases} \quad (39)$$

It is seen that Eq. (39) has not a non-trivial solution. In the third case $\alpha_0 = 1$ and $\alpha_1 = -1$, Eq. (36) could be written as:

$$\begin{cases} 2e^{-t_0'} - e^{-2t_0'} + 4e^{-t_1'} - 2e^{-2t_1'} = 1 \\ e^{-t_0'} - e^{-2t_0'} + 2e^{-t_1'} - 2e^{-2t_1'} = 0 \end{cases} \quad (40)$$

Then

$$\begin{cases} 2e^{-t_1'} + e^{-t_0'} = 1 \\ 2e^{-2t_1'} + e^{-2t_0'} = 1 \end{cases} \quad (41)$$

It is clear that Eq. (41) would have not a real solution. Finally, in the last case $\alpha_0 = \alpha_1 = -1$, Eq. (36) could be simplified as:

$$\begin{cases} 2e^{-t_0'} - e^{-2t_0'} = \frac{1}{3} \\ -2e^{-t_0'} + 2e^{-2t_0'} = 0 \end{cases} \quad (42)$$

It can be also shown that there would not be any real solution for Eq. (42). Hence the final time is calculated as $t_f = t_0' = 1.8663$ seconds and also the switching time is determined as $t_1 = t_0' - t_1' = Ln\left(\sqrt{3}+3\right) = 1.5544$ seconds. Then the optimal control signal $u(t)$ would be represented as follows:

$$u(t) = \begin{cases} -1 & 0 \leq t < 1.5544 \\ +1 & 1.5544 \leq t < 1.8663 \end{cases} \quad (43)$$

Therefore the signal $u(t)$ would be a min-time optimal control in the dynamical system (33).

Example 2: Consider the following second order system:

$$\dot{x}=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}x+\begin{bmatrix}0\\1\end{bmatrix}u \tag{44}$$

The initial and final conditions are selected as $x(0)=[1 \quad 0]^T$ and $x(t_f)=[0 \quad 0]^T$. The poles of the LTI system (44) are complex. Hence the algorithm 1 would be used in this example. The vectors β_0, β_1 and x_f' are determined as follows:

$$\beta_0=\frac{2-\alpha_0}{2}\begin{bmatrix}1\\0\end{bmatrix},\ \beta_1=-\frac{\alpha_1-\alpha_0}{2}\begin{bmatrix}1\\0\end{bmatrix},\ x_f'=-\frac{1}{2}\alpha_1\begin{bmatrix}1\\0\end{bmatrix}$$

The state transition matrix $\Phi(t)$ may be determined as follows:

$$\Phi(t)=\begin{bmatrix}e^{-t}(\cos(t)+\sin(t)) & e^{-t}\sin(t)\\ -2e^{-t}\sin(t) & e^{-t}(\cos(t)-\sin(t))\end{bmatrix}$$

The min-time control problem can be solved via the following equation:

$$x_f'=\Phi(t_0')\beta_0+\Phi(t_1')\beta_1$$

where: $t_0'=t_2\geq 0$ and $t_1'=t_2-t_1\geq 0$. Then

$$\alpha_1\begin{bmatrix}1\\0\end{bmatrix}=(\alpha_0-2)\Phi(t_0')\begin{bmatrix}1\\0\end{bmatrix}+(\alpha_1-\alpha_0)\Phi(t_1')\begin{bmatrix}1\\0\end{bmatrix} \tag{45}$$

It can be rewritten as follows:

$$\alpha_1\begin{bmatrix}1\\0\end{bmatrix}=(\alpha_0-2)\begin{bmatrix}e^{-t_0'}(\cos(t_0')+\sin(t_0'))\\ -2e^{-t_0'}\sin(t_0')\end{bmatrix}+ (\alpha_1-\alpha_0)\begin{bmatrix}e^{-t_1'}(\cos(t_1')+\sin(t_1'))\\ -2e^{-t_1'}\sin(t_1')\end{bmatrix} \tag{46}$$

Eq. (46) may be simplified as follows:

$$\begin{cases}(\alpha_0-2)e^{-t_0'}\cos(t_0')+\\(\alpha_1-\alpha_0)e^{-t_1'}\cos(t_1')=\alpha_1\\(\alpha_0-2)e^{-t_0'}\sin(t_0')+\\(\alpha_1-\alpha_0)e^{-t_1'}\sin(t_1')=0\end{cases} \tag{47}$$

In similar way, Eq. (47) has a solution in condition which $\alpha_0=-1$ and $\alpha_1=1$. It can be checked that Eq. (47) has not a feasible solution in the other cases. Then the solution of Eq. (47) is computed as $t_0'=1.6026$ and $t_1'=0.5518$. Therefore the final time is calculated as $t_f=t_0'=1.6026$ seconds and also the switching time is found as $t_1=t_0'-t_1'=1.0507$ seconds.

Example 3: Consider the following third order LTI system:

$$\begin{cases}\dot{x_1}=x_2\\ \dot{x_2}=x_3\\ \dot{x_3}=u\end{cases} \tag{48}$$

The matrix A is not invertible in the dynamical system (48). Hence the algorithm 2 can be used in this example. The initial and final conditions are chosen as $x(0)=[1 \quad -1 \quad 1]^T$ and $x(t_f)=0$. In the algorithm 2, we have:

$$t_0'=t_n\,,t_1'=t_n-t_1\,,t_2'=t_n-t_2\,,\gamma_0=\alpha_0,$$

$$\gamma_0=\alpha_0,\gamma_1=\alpha_1-\alpha_0,\ \gamma_2=\alpha_2-\alpha_1,\tilde{x}_f=x_f-\Phi(t_f)x_0$$

The state transition matrix $\Phi(t)$ may be determined as follows:

$$\Phi(t)=\begin{bmatrix}1 & t & \frac{1}{2}t^2\\ 0 & 1 & t\\ 0 & 0 & 1\end{bmatrix}$$

Then the vector $\Omega(t)$ is computed as:

$$\Omega(t)=\int_{t_0}^{t}\Phi(\tau)Bd\tau=\begin{bmatrix}\frac{1}{6}t^3\\ \frac{1}{2}t^2\\ t\end{bmatrix}$$

Eq. (22) may be rewritten as the following:

$$\gamma_0\Omega(t_0')+\gamma_1\Omega(t_1')+\gamma_2\Omega(t_2')=\tilde{x}_f \tag{49}$$

Then

$$\alpha_0\begin{bmatrix}\frac{1}{6}{t_0'}^3\\ \frac{1}{2}{t_0'}^2\\ t_0'\end{bmatrix}+(\alpha_1-\alpha_0)\begin{bmatrix}\frac{1}{6}{t_1'}^3\\ \frac{1}{2}{t_1'}^2\\ t_1'\end{bmatrix}+ (\alpha_2-\alpha_1)\begin{bmatrix}\frac{1}{6}{t_2'}^3\\ \frac{1}{2}{t_2'}^2\\ t_2'\end{bmatrix}+\begin{bmatrix}1-t_f+\frac{1}{2}{t_f}^2\\ -1+t_f\\ 1\end{bmatrix}=0 \tag{50}$$

It can be simplified as follows:

$$\begin{cases} \alpha_0 {t'_0}^3 + (\alpha_1 - \alpha_0){t'_1}^3 + \\ (\alpha_2 - \alpha_1){t'_2}^3 + 6 - 6t_f + 3{t_f}^2 = 0 \\ \alpha_0 {t'_0}^2 + (\alpha_1 - \alpha_0){t'_1}^2 + \\ (\alpha_2 - \alpha_1){t'_2}^2 - 2 + 2t_f = 0 \\ \alpha_0 t'_0 + (\alpha_1 - \alpha_0)t'_1 + \\ (\alpha_2 - \alpha_1)t'_2 + 1 = 0 \end{cases} \quad (51)$$

Eq. (51) would have a solution when $\alpha_0 = -1$, $\alpha_1 = 1$ and $\alpha_2 = -1$. It is not hard to check that Eq. (51) has not a feasible solution in the other cases. The solution of Eq. (51) is calculated as $t'_0 = 2.3735$, $t'_1 = 1.3942$ and $t'_2 = 0.7074$. Therefore the final time is calculated as $t_f = t'_0 = 1.6026$ seconds and also the switching times are determined as $t_1 = t'_0 - t'_1 = 0.9793$ and $t_2 = t'_0 - t'_2 = 1.6661$ seconds.

5-Conclusion

The exact solution of the min-time problem is analytically investigated in the constrained continuous-time dynamical system. The min-time control problem is firstly formulated in the LTI system with input constraint. Then a systematic procedure based on the state transition matrix is proposed to determine the min-time control signal explicitly. The min-time control signal would be explicitly determined by computation of the switching times and also some other constants. Finally the optimal control input is represented as a switched signal. Such a control policy is used in a second order system to show the effectiveness of the proposed method.

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