On Infinitesimal Conformal Transformations of the Tangent Bundles with the Generalized Metric

E. Peyghani, A. Tayebi

ABSTRACT

Let \( (M, g) \) be an n-dimensional Riemannian manifold, and \( TM \) be its tangent bundle with the lift metric \( G \). Then every infinitesimal fiber-preserving conformal transformation \( X \) induces an infinitesimal homothetic transformation \( V \) on \( M \). Furthermore, the correspondence \( X \to V \) gives a homomorphism of the Lie algebra of infinitesimal fiber-preserving conformal transformations on \( TM \) onto the Lie algebra of infinitesimal homothetic transformations on \( M \), and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infinitesimal isometries of \( M \).

KEYWORDS

Infinitesimal conformal transformation, homothetic transformation, Lagrange metric, isometry

1. INTRODUCTION

In the present paper everything will be always discussed in the \( C^\infty \) category, and Riemannian manifolds will be assumed to be connected and \( \dim M > 1 \).

Let \( M \) be an n-dimensional Riemannian manifold with a metric \( g \) and \( \phi \) be a transformation on \( M \). Then \( \phi \) is called a conformal transformation if it preserves the angles. Let \( V \) be a vector field on \( M \) and \( \{ \phi_t \} \) be the local one-parameter group of local transformations on \( M \) generated by \( V \). Then \( V \) is called an infinitesimal conformal transformation, if each \( \phi_t \) is a local conformal transformation of \( M \). It is well known that \( V \) is an infinitesimal conformal transformation if and only if there exists a scalar function \( \Omega \) on \( M \) such that

\[
\mathcal{L}_V g = 2\Omega g
\]

(1)

where \( \mathcal{L}_V \) denotes the Lie derivation with respect to the vector field \( V \), especially \( V \) is called an infinitesimal homothetic one when \( \Omega \) is constant [7].

In the presence of a chart \( x = (x^i)_{1 \leq i \leq n} \), the equation (1) reduce to

\[
\nabla_i v_j + \nabla_j v_i = 2\Omega g_{ij}
\]

(2)

where

\[
g = g_{ij} dx^i \otimes dx^j, \quad \nabla_i v_j = \frac{\partial v_j}{\partial x^i} - \Gamma^k_{ij} v_k
\]

and

\[
V = v^i \frac{\partial}{\partial x^i}.
\]

Raising \( j \) and contracting with \( i \) in (2) it is easily seen that

\[
\nabla_i v^i + \nabla^j v_j = 2n\Omega
\]

Hence

\[
\Omega = \frac{1}{n} \text{div}(V)
\]

where \( n = \dim M \).

Example. Let \( n \geq 3 \). On the n-dimensional Euclidean space, which is simply the manifold \( \mathbb{R}^n \) with the Riemannian metric tensor field \( g = g_{ij} dx^i \otimes dx^j \) where \( g_{ij} \) is a constant for each \( 1 \leq i, j \leq n \), equation (2) reduces to

\[
\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j} = 2\Omega g_{ij}
\]

(3)

from which it can be deduced that \( \Omega \) must be of the form

\[
\Omega = b_k x^k + c
\]

for some constants \( b_k, c \in \mathbb{R} \) and

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1 Corresponding Author, E. Peyghani is with the Faculty of Science, Department of Mathematics, Arak University, Arak, Iran (e-mail: epeyghan@gmail.com).

ii A. Tayebi is with the Faculty of Science, Department of Mathematics, Qom University, Qom, Iran (e-mail: akbar.tayebi@gmail.com).
\[ v_i = A_i + x^k H_{ki} + \frac{1}{2} x^k x^l (a_k g_{il} + a_l g_{ki} - a_i g_{kl}) \]

where \( A_i, H_{ki} \in \mathbb{R} \) are constants and \( H_{ki} + H_{ik} = 2c g_{ki} \). We notice that this conformal vector field is homothetic if and only if the vector \( a = (a^i)^{\text{signature}} \) vanishes.

Let \( TM \) be the tangent space of \( M \), and let \( \Phi \) be a transformation of \( TM \). Then \( \Phi \) is called a fiber-preserving transformation, if it preserves the fibers. Let \( X \) be a vector field on \( TM \), and let us consider the local one-parameter group \( \{ \Phi_{\tau} \} \) of local transformations of \( TM \) generated by \( X \). Then \( X \) is called an infinitesimal fiber-preserving transformation on \( TM \), if \( \Phi_{\tau} \) is a local fiber-preserving transformation of \( TM \). Clearly an infinitesimal fiber-preserving transformation on \( TM \) induces an infinitesimal transformation in the base space \( M \) [4]. Let \( \overline{g} \) be a (pseudo)-Riemannian metric of \( TM \). An infinitesimal fiber-preserving transformation \( X \) on \( TM \) is said to be an infinitesimal fiber-preserving conformal transformation, if there exists a scalar function \( \overline{\rho} \) on \( TM \) such that \( \mathcal{L}_X \overline{g} = 2 \overline{\rho} \overline{g} \), where \( \mathcal{L}_X \) denotes the Lie derivation with respect to \( X \) [7].

The purpose of the present paper is to prove the following theorem:

**Theorem.** Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold, and \( TM \) be its tangent space with the lift metric \( G \). Then every infinitesimal fiber-preserving conformal transformation \( X \) of \( TM \) induces an infinitesimal homothetic transformation \( V \) on \( M \). Furthermore, the correspondence \( X \rightarrow V \) gives a homomorphism of the Lie algebra of infinitesimal fiber-preserving conformal transformations on \( TM \) onto the Lie algebra of infinitesimal homothetic transformations on \( M \), and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infinitesimal isometries of \( M \).

2. **Generalized Metric \( G \)**

Let \((M, g)\) be a (pseudo)-Riemannian manifold and \( \nabla \) be its Levi-Civita connection. In a local chart \((U, (x'))\) we set \( g'_{ij} = g(\partial_i, \partial_j) \), where \( \partial_i := \frac{\partial}{\partial x^i} \)

and we denote by \( \Gamma^{\prime}_{jk} \) the Christoffel symbols of \( \nabla \). Let \((x', y') \equiv (x, y)\) be the local coordinates on the manifold \( TM \) projected on \( M \) by \( \pi \). The indices \( i, j, k, \ldots \) will run from 1 to \( n = \text{dim} M \).

The functions \( N^i_j(x, y) := \Gamma^i_{jk}(x) y^k \) are the local coefficients of a nonlinear connection, that is the local vector fields

\[ \delta_{\tau} = \partial_{\tau} - N^i_j(x, y) \partial_i, \]

where \( \partial_{\tau} = \frac{\partial}{\partial \tau} \) span distribution on \( TM \) called horizontal which is supplementary to the vertical distribution \( u \rightarrow V_u TM = \ker(\tau_u) \), where \( u \in TM \).

Let us denote by \( u \rightarrow H_u TM \) the horizontal distribution and let \{\( \delta_i, \partial_{\tau} \)\} be the basis adapted to the decomposition

\[ T_u TM = H_u TM \oplus V_u TM \]

where \( u \in TM \). The basis dual of it is \{\( dx^i, \delta y^i \)\} with \( \delta y^i = dy^i + N^i_j(x, y)dx^j \).

We can easily prove the following lemma:

**Lemma 1.** The Lie brackets satisfy the following:

\[ [\delta_i, \delta_j] = y^r K^m_{ij} \partial_m, \]

\[ [\delta_i, \partial_{\tau}] = \Gamma^{\tau}_{ij} \partial_m, \]

\[ [\partial_{\tau}, \partial_{\tau}] = 0, \]

where \( K^m_{ij} \) denote the components of the curvature tensor of \( M \).

The metric \( II + III \) on \( TM \) is as follows:

\[ II + III = 2g_{ij}(x)dx^i \delta y^j + g_{ij}(x)\delta y^i \delta y^j. \]

If in the term of \( G \) one replaces \( g_{ij}(x) \) with the components \( h_{ij}(x, y) \) of a generalized Lagrange metric(5)) one gets a metric

\[ G(x, y) = 2h_{ij}(x, y)dx^i \delta y^j + h_{ij}(x, y)\delta y^i \delta y^j. \]

In particular, \( h_{ij}(x, y) \) could be a deformation of \( g_{ij}(x) \), a case studied by M. Anastasiei in [3].

In this paper, we are concerning with the metric \( G \) in the case when \( h_{ij}(x, y) \) is the following special deformation of \( g_{ij}(x) \)

\[ h_{ij}(x, y) = a(L^2)g_{ij}(x), \]
where \( L^2 = g_{ij}(x)y^iy^j \), \( y_i = g_{ij}(x)y^j \) and 
\( a : \text{Im}(L^2) \subseteq R_+ \rightarrow R_+ \) with \( a > 0 \).

3. INFINITESIMAL CONFORMAL TRANSFORMATION

Let \( X \) be an infinitesimal fiber-preserving transformation on \( TM \) and \((v^h, \tilde{\gamma}^h)\) be the components of \( X \) with respect to the adapted frame \( \{dx^h, \delta y^h\} \).

Then \( X \) is fiber-preserving if and only if \( \tilde{\gamma}^h \) depend only on variables \((x^h)\). Clearly \( X \) induces an infinitesimal transformation \( V \) with the components \( \tilde{\gamma}^h \) in the base space \( M \) [4]. We have the following lemma:

**Lemma 2.** The Lie derivative of the adapted frame and the dual basis are given as follows:

1. \[ \mathbf{L}_X \delta_a = -\partial_a \nu^\beta \delta_a + \gamma^h_a \gamma^\nu_{h\nu} - \delta_a \nu^\beta \delta_a, \]
2. \[ \mathbf{L}_X \tilde{\gamma}_a = \nu^\beta \delta_a \nu^\beta - \tilde{\gamma}_a, \]
3. \[ \mathbf{L}_X \nu_a = \tilde{\gamma}^h_a \nu^h + \delta_a \nu^\beta \delta_a, \]
4. \[ \mathbf{L}_X \mathbf{y}^h = -\gamma^h_a \mathbf{y}^a - \delta_a \nu^\beta \delta_a \mathbf{y}^h, \]
5. \[ \mathbf{L}_X \delta y^h = -\partial_a \nu^\beta \delta_a \delta y^h. \]

**Proof.** Proof of this lemma, is similar to proof of the Proposition 2.2 of Yamauchi [7].

**Lemma 3.** The Lie derivative \( \mathbf{L}_X G \) is in the following form:

\[ \mathbf{L}_X G = -2a(L^2) \left( \nu^\beta \mathbf{y}^a \gamma^h_a \mathbf{y}^a - \delta_a \nu^\beta \right) \mathbf{y}^h \mathbf{y}^j \]
\[ + 2a(L^2) \left( \nu^\beta \mathbf{y}^a \gamma^h_a \mathbf{y}^a - \delta_a \nu^\beta \delta_a \mathbf{y}^h \mathbf{y}^j \right) \]
\[ + 2a(L^2) \nu^\beta \delta_a \mathbf{y}^h \mathbf{y}^j \]
\[ + 2a(L^2) \left( \nu^\beta \mathbf{y}^a \gamma^h_a \mathbf{y}^a - \delta_a \nu^\beta \delta_a \mathbf{y}^h \mathbf{y}^j \right) \]
where \( \nu^\beta = \tilde{\gamma}^h_a \mathbf{y}^a \).

**Proof.** From the definition of Lie derivative we have:
Proposition 4. The vector field V with the components \((v^h)\) is an infinitesimal conformal transformation on \(M\).

Proof. By replace \(i\) and \(j\) in (10) and addition new relation with (10), we get
\[
2\mathcal{L}_V g_{ij} - g_{im} \nabla_j v^m + g_{jm} \nabla_i v^m + g_{jm} \nabla_i (v^m) - g_{mi} \{y^h \psi^K_{mb} - \nabla^m \Gamma^i_{bi} - \delta_i (v^m)\} - g_{mi} \{y^h \psi^K_{mb} - \nabla^m \Gamma^i_{bi} - \delta_i (v^m)\} = 4\bar{\Omega} g_{ij},
\]
By attention to (7), (11) and equation \(\nabla g_{ij} = \nabla + \nabla\), we have
\[
\mathcal{L}_V g_{ij} = 4\bar{\Omega} g_{ij}.
\]
This shows the scalar function \(\bar{\Omega}\) on \(TM\) depends only on the variables \((x^h)\) with respect to the induced coordinates \((x^h, y^h)\), thus we can regard \(\bar{\Omega}\) as a function on \(M\), and \(V\) is an infinitesimal conformal transformation on \(M\).

Proposition 5. The vertical components \((v^h)\) of \(X\) can be written as the following form:
\[
v^h = y^i A^h_i + B^h,
\]
where \(A^h_i\) and \(B^h\) are the components of a certain \((1,1)\) tensor field \(A\) and a certain contravariant vector field \(B\) on \(M\), respectively.

Proof. By derivation of (11) respect \(\partial_r\), we get
\[
g_{mu} \partial_r \partial_T(v^m) + g_{mo} \partial_r \partial_T(v^m) = 2g_{ij} \partial_r (\bar{\Omega}).
\]
Since the scalar function \(\bar{\Omega}\) on \(TM\) depends only on the variables \((x^h)\), thus we have \(\partial_r (\bar{\Omega}) = 0\). Therefore we get
\[
g_{mu} \partial_r \partial_T(v^m) + g_{mo} \partial_r \partial_T(v^m) = 0.
\]
Then we have
\[
g_{mu} \partial_r \partial_T(v^m) = -g_{mo} \partial_r \partial_T(v^m) = -\partial_r (g_{mu} \partial_T(v^m)) = -\partial_r (-g_{mu} \partial_T(v^m) + 2\bar{\Omega} g_{ij})
\]
\[
= g_{mu} \partial_r \partial_T(v^m) = \partial_T(g_{mu} \partial_r (v^m)) = \partial_T (-g_{mu} \partial_r (v^m) + 2\bar{\Omega} g_{ij})
\]
\[
= -g_{mu} \partial_r \partial_T(v^m) = -g_{mu} \partial_r \partial_T(v^m).
\]
which implies that \(g_{mu} \partial_r \partial_T(v^m) = 0\). This shows that \(\partial_T(v^m)\) depends only on the variables \((x^h)\). Hence \(v^h\) can be written as \(v^h = y^i A^h_i + B^h\), where \(A^h_i\) and \(B^h\) are certain function on \(M\). The coordinate transformation rule implies \(A^h_i\) and \(B^h\) are the components of a certain \((1,1)\) tensor field \(A\) and a certain contravariant vector field \(B\). □

Proposition 6. The vector field \(B = (B^h)\) is an infinitesimal isometry on \(M\).

Proof. Substituting equation (12) into the equation (10) and (11), then by Proposition 4, we can get
\[
A_{ij} - \nabla_j v_i + \nabla_i B_j = 0,
\]
\[
\nabla j A^h_i + K^h_{ij} v' = 0,
\]
\[
A_{ij} + A_{ji} = 2\Omega g_{ij}.
\]
From equation (13), (15) and Proposition 4, we have
\[
\nabla_j g_{ij} = \nabla_j B_i + \nabla_i B_j = 0.
\]
Thus the vector field \(B\) is an infinitesimal isometry on \(M\). □

Proposition 7. The scalar function \(\bar{\Omega}\) on \(M\) is a constant function.

Proof. From equation (11), (12), we have
\[
2\nabla_k (\Omega g_{ij}) = \nabla_k (A_{ij} + A_{ji})
\]
\[
= -K_{abj} v^a - K_{abj} v^a = 0.
\]
Thus the scalar function \(\bar{\Omega}\) on \(M\) is constant. □

By proposition 7, the vector field \(V\) on \(M\) become infinitesimal homothetic transformation.

Conversely, let \(V = (v^h)\) be an infinitesimal homothetic transformation on \(M\) that is, there exists a constant \(c\) such that \(\xi g_{ij} = 2cg_{ij}\). Then we define the vector field \(X\) on \(TM\) as follows
\[
X = v^h X_h + y^a \nabla v^h X^a.
\]
Proposition 8. The vector field \(X\) on \(TM\) defined above is an infinitesimal conformal transformation.

Proof. From lemma 3 we have:
\[\mathcal{L}_x G = -2a(L^2)g_{jm} \{ y^b v^c K_{icb}^m - y^a \nabla_a v^b \Gamma_{bi}^m \\
- \delta_i(y^a \nabla_a v^b) \} dx^j \delta y^i \\
+ 2a(L^2) \{ 2(\varphi + c) g_{ij} dx^j \delta y^i \\
+ 2a(L^2)(\varphi + c) g_{ij} \delta y^i \delta y^j \} \]

Thus we have \(\mathcal{L}_x G = 2\varOmega G\). This shows the vector field \(X\) on \(TM\) is an infinitesimal conformal transformation. \(\Box\)

**Proof of Theorem.** Summing up proposition 1 to proposition 5, it is clear that the correspondence \(X \rightarrow V\) gives a homomorphism of the Lie algebra of infinitesimal fiber-preserving conformal transformations on \(TM\) onto the Lie algebra of infinitesimal homothetic transformations on \(M\), and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infinitesimal isometries of \(M\).

### 4. REFERENCES


