On Infinitesimal Conformal Transformations of the Tangent Bundles with the Generalized Metric

E. Peyghan[i], A. Tayebi[ii]

Abstract

Let (M, g) be an n-dimensional Riemannian manifold, and TM be its tangent bundle with the lift metric G. Then every infinitesimal fiber-preserving conformal transformation X induces an infinitesimal homothetic transformation V on M. Furthermore, the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fiber-preserving conformal transformations on TM onto the Lie algebra of infinitesimal homothetic transformations on M, and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infinitesimal isometries of M.

Keywords

Infinitesimal conformal transformation, homothetic transformation, Lagrange metric, isometry

1. Introduction

In the present paper everything will be always discussed in the C^∞ category, and Riemannian manifolds will be assumed to be connected and $dimM > 1$.

Let M be an n-dimensional Riemannian manifold with a metric g and ϕ be a transformation on M. Then ϕ is called a *conformal* transformation if it preserves the angles. Let V be a vector field on M and $\{\varphi_t\}$ be the local one-parameter group of local transformations on M generated by V. Then V is called an infinitesimal conformal transformation, if each φ_t is a local conformal transformation of M. It is well known that V is an infinitesimal conformal transformation if and only if there exists a scalar function Ω on M such that

$$\pounds_V g = 2\Omega g \qquad (1)$$

where $\pounds_V$ denotes the Lie derivation with respect to the vector field V, especially V is called an infinitesimal homothetic one when Ω is constant [7].

In the presence of a chart $x = (x^i)_{1\le i\le n}$, the equation (1) reduce to

$$\nabla_i v_j + \nabla_j v_i = 2\Omega g_{ij} \qquad (2)$$

where

$$g = g_{ij} dx^i \otimes dx^j, \quad \nabla_i v_j = \frac{\partial v_j}{\partial x^i} - \Gamma_{ij}{}^k v_k$$

and

$$V = v^i \frac{\partial}{\partial x^i}.$$

Raising j and contracting with i in (2) it is easily seen that

$$\nabla_i v^i + \nabla_j v^j = 2n\Omega$$

Hence

$$\Omega = \frac{1}{n} div(V)$$

where $n = \dim M$.

Example. Let $n \ge 3$. On the n-dimensional Euclidean space, which is simply the manifold R^n with the Riemannian metric tensor field $g = g_{ij} dx^i \otimes dx^j$ where g_{ij} is a constant for each $1 \le i, j \le n$, equation (2) reduces to

$$\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j} = 2\Omega g_{ij} \qquad (3)$$

from which it can be deduced that Ω must be of the form $\Omega = b_k x^k + c$ for some constants $b_k, c \in R$ and

i Corresponding Author, E. Peyghan is with the Faculty of Science, Department of Mathematics, Arak University, Arak, Iran (e-mail: epeyghan@gmail.com).

ii A. Tayebi is with the Faculty of Science, Department of Mathematics, Qom University, Qom, Iran (e-mail: akbar.tayebi@gmail.com).

$$v_i = A_i + x^k H_{ki} + \frac{1}{2} x^k x^r (a_k g_{ri} + a_r g_{ki} - a_i g_{kr})$$

where $A_i, H_{ki} \in R$ are constants and $H_{ki} + H_{ik} = 2cg_{ki}$. We notice that this conformal vector field is homothetic if and only if the vector $a = (a^i)_{1 \le i \le n}$ vanishes.

Let TM be the tangent space of M, and let Φ be a transformation of TM. Then Φ is called a fiber-preserving transformation, if it preserves the fibers. Let X be a vector field on TM, and let us consider the local one-parameter group $\{\Phi_t\}$ of local transformations of TM generated by X. Then X is called an infinitesimal fiber-preserving transformation on TM, if each Φ_t is a local fiber-preserving transformation of TM. Clearly an infinitesimal fiber-preserving transformation on TM induces an infinitesimal transformation in the base space M [4]. Let $\bar{g}$ be a (pseudo)-Riemannian metric of TM. An infinitesimal fiber-preserving transformation X on TM is said to be an infinitesimal fiber-preserving conformal transformation, if there exists a scalar function $\bar{\rho}$ on TM such that $\pounds_X \bar{g} = 2\bar{\rho}\bar{g}$, where $\pounds_X$ denotes the Lie derivation with respect to X [7].

The purpose of the present paper is to prove the following theorem:

Theorem . *Let (M, g) be an n-dimensional Riemannian manifold, and TM be its tangent space with the lift metric G. Then every infinitesimal fiber-preserving conformal transformation X of TM induces an infinitesimal homothetic transformation V on M. Furthermore, the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fiber-preserving conformal transformations on TM onto the Lie algebra of infinitesimal homothetic transformations on M, and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infinitesimal isometries of M.*

2. Generalized metric G

Let (M, g) be a (pseudo)-Riemannian manifold and ∇ be its Levi-Civita connection. In a local chart $(U, (x^i))$ we set $g_{ij} = g(\partial_i, \partial_j)$, where $\partial_i := \frac{\partial}{\partial x^i}$ and we denote by Γ^i_{jk} the Christoffel symbols of ∇. Let $(x^i, y^i) \equiv (x, y)$ be the local coordinates on the manifold TM projected on M by π. The indices $i, j, k, ...$ will run from 1 to $n = dimM$.

The functions $N^i_j(x, y) := \Gamma^i_{jk}(x) y^k$ are the local coefficients of a nonlinear connection, that is the local vector fields

$$\delta_i = \partial_i - N^k_i(x, y)\partial_{\bar{k}},$$

where $\partial_{\bar{k}} = \frac{\partial}{\partial y^k}$ span distribution on TM called horizontal which is supplementary to the vertical distribution $u \to V_u TM = ker(\tau_*)_u$ where $u \in TM$. Let us denote by $u \to H_u TM$ the horizontal distribution and let $\{\delta_i, \partial_{\bar{i}}\}$ be the basis adapted to the decomposition

$$T_u TM = H_u TM \oplus V_u TM$$

where $u \in TM$. The basis dual of it is $\{dx^i, \delta y^i\}$ with $\delta y^i = dy^i + N^i_k(x, y)dx^k$.

We can easily prove the following lemma:

Lemma 1. *The Lie brackets satisfy the following*:

$$[\delta_i, \delta_j] = y^r K^m_{jir} \partial_{\bar{m}},$$
$$[\delta_i, \partial_{\bar{j}}] = \Gamma^m_{ji} \partial_{\bar{m}},$$
$$[\partial_{\bar{i}}, \partial_{\bar{j}}] = 0,$$

where $K_{jir}{}^m$ *denote the components of the curvature tensor of* M.

The metric $II + III$ on TM is as follows:

$$II + III = 2g_{ij}(x)dx^i \delta y^j + g_{ij}(x)\delta y^i \delta y^j.$$

If in the term of G one replaces $g_{ij}(x)$ with the components $h_{ij}(x, y)$ of a generalized Lagrange metric([5]) one gets a metric

$$G(x, y) = 2h_{ij}(x, y)dx^i \delta y^j + h_{ij}(x, y)\delta y^i \delta y^j.$$

In particular, $h_{ij}(x, y)$ could be a deformation of $g_{ij}(x)$, a case studied by M. Anastasiei in [3].

In this paper, we are concerning with the metric G in the case when $h_{ij}(x, y)$ is the following special deformation of $g_{ij}(x)$

$$h_{ij}(x, y) = a(L^2) g_{ij}(x),$$

where $L^2 = g_{ij}(x)y^i y^j$, $y_i = g_{ij}(x)y^j$ and $a : Im(L^2) \subseteq R_+ \longrightarrow R_+$ with $a > 0$.

3. Infinitesimal Conformal Transformation

Let X be an infinitesimal fiber-preserving transformation on TM and $(v^h, v^{\bar{h}})$ be the components of X with respect to the adapted frame $\{dx^h, \delta y^h\}$. Then X is fiber-preserving if and only if v^h depend only on variables (x^h). Clearly X induces an infinitesimal transformation V with the components v^h in the base space M [4]. We have the following lemma:

Lemma 2. *The Lie derivative of the adapted frame and the dual basis are given as follows:*

(1) $\pounds_X \delta_h = -\partial_h v^a \delta_a$

$$+\{y^b v^c K^a_{hcb} - v^{\bar{b}} \Gamma^a_{bh} - \delta_h (v^{\bar{a}})\}\delta_{\bar{a}},$$

(2) $\pounds_X \partial_{\bar{h}} = \{v^b \Gamma^a_{hb} - \delta_{\bar{h}}(v^{\bar{a}})\}\delta_{\bar{a}}$,

(3) $\pounds_X dx^h = \partial_m v^h dx^m$,

(4) $\pounds_X \delta y^h = -\{y^b v^c K^h_{mcb} - v^{\bar{b}} \Gamma^h_{bm} - \delta_m (v^{\bar{h}})\}dx^m$

$$-\{v^b \Gamma^h_{mb} - \delta_{\bar{m}}(v^{\bar{h}})\}\delta y^m.$$

Proof. Proof of this lemma, is similar to proof of the Proposition 2.2 of Yamauchi [7]. □

Lemma 3. *The Lie derivative* $\pounds_X G$ *is in the following form:*

$$\begin{aligned}\pounds_X G = &-2a(L^2) g_{im}\{y^b v^c K^m_{jcb} - v^{\bar{b}} \Gamma^m_{bj} \\ &- \delta_j (v^{\bar{m}})\}dx^i dx^j \\ &+ 2a(L^2)\{2\bar{\varphi} g_{ij} + \pounds_V g_{ij} \\ &- g_{im}\nabla_j v^m + g_{im}\partial_{\bar{j}}(v^{\bar{m}}) \\ &- g_{mj} y^b v^c K^m_{icb} + g_{mj} v^{\bar{b}} \Gamma^m_{bi} \\ &+ g_{mj}\delta_i (v^{\bar{m}})\}dx^i \delta y^j \\ &+ 2a(L^2)(\bar{\varphi} g_{ij} + g_{mi}\partial_{\bar{j}}(v^{\bar{m}}))\delta y^i \delta y^j\end{aligned}$$

where $\bar{\varphi} = v^{\bar{h}} y_h \frac{a'(L^2)}{a(L^2)}$.

Proof. From the definition of Lie derivative we have:

$$\begin{aligned}\pounds_X G = &\pounds_X (a(L^2))(2g_{ij} dx^i \delta y^j + g_{ij}\delta y^i \delta y^j) \\ &+ a(L^2)\pounds_X (2g_{ij} dx^i \delta y^j \\ &+ g_{ij}\delta y^i \delta y^j)\end{aligned} \tag{4}$$

From lemma 2 we conclude the following result:

$$\begin{aligned}&\pounds_X (2g_{ij} dx^i \delta y^j + g_{ij}\delta y^i \delta y^j) = \\ &-2g_{im}\{y^b v^c K^m_{jcb} - v^{\bar{b}} \Gamma^m_{bj} - \delta_j (v^{\bar{m}})\}dx^i dx^j \\ &+2\{\pounds_V g_{ij} - g_{im}\nabla_j v^m + g_{im}\partial_{\bar{j}}(v^{\bar{m}}) \\ &-g_{mj} y^b v^c K^m_{icb} + g_{mj} v^{\bar{b}} \Gamma^m_{bi} + g_{mj}\delta_i (v^{\bar{m}})\}dx^i \delta y^j \\ &+2g_{mi}\partial_{\bar{j}}(v^{\bar{m}})\delta y^i \delta y^j\end{aligned} \tag{5}$$

Since $\delta_h (L^2) = 0$ and $\partial_{\bar{h}}(L^2) = 2y_h$, we have:

$$\begin{aligned}\pounds_X (a(L^2)) &= X(a(L^2)) \\ &= v^h \delta_h (a(L^2)) + v^{\bar{h}} \partial_{\bar{h}}(a(L^2)) \\ &= 2v^{\bar{h}} y_h a'(L^2)\end{aligned} \tag{6}$$

By taking (5) and (6) in (4), we have the proof. □

Let X be an infinitesimal fiber-preserving conformal transformation on TM with respect to metric G, that is, there exists a scalar function $\bar{\rho}$ on TM such that

$$\pounds_X G = 2\bar{\rho} G$$

Then from lemma 3, we have

$$\begin{aligned}&g_{im}\{y^b v^c K^m_{jcb} - v^{\bar{b}} \Gamma^m_{bj} - \delta_j (v^{\bar{m}})\} \\ &+g_{jm}\{y^b v^c K^m_{icb} - v^{\bar{b}} \Gamma^m_{bi} - \delta_i (v^{\bar{m}})\} = 0,\end{aligned} \tag{7}$$

$$\begin{aligned}&2\bar{\varphi} g_{ij} + \pounds_V g_{ij} - g_{im}\nabla_j v^m + g_{im}\partial_{\bar{j}}(v^{\bar{m}}) \\ &-g_{mj}\{y^b v^c K^m_{icb} - v^{\bar{b}} \Gamma^m_{bi} - \delta_i (v^{\bar{m}})\} = 2\bar{\rho} g_{ij},\end{aligned} \tag{8}$$

$$2\bar{\varphi} g_{ij} + g_{mi}\partial_{\bar{j}}(v^{\bar{m}}) + g_{mj}\partial_{\bar{i}}(v^{\bar{m}}) = 2\bar{\rho} g_{ij}. \tag{9}$$

Let $\bar{\Omega} = \bar{\rho} - \bar{\varphi}$. From (8) and (9) we conclude following relations:

$$\begin{aligned}&\pounds_V g_{ij} - g_{im}\nabla_j v^m + g_{im}\partial_{\bar{j}}(v^{\bar{m}}) \\ &-g_{mj}\{y^b v^c K^m_{icb} - v^{\bar{b}} \Gamma^m_{bi} - \delta_i (v^{\bar{m}})\} = 2\bar{\Omega} g_{ij},\end{aligned} \tag{10}$$

$$g_{mi}\partial_{\bar{j}}(v^{\bar{m}}) + g_{mj}\partial_{\bar{i}}(v^{\bar{m}}) = 2\bar{\Omega} g_{ij}. \tag{11}$$

Proposition 4. *The vector field* V *with the components* (v^h) *is an infinitesimal conformal transformation on* M .

Proof. By replace i and j in (10) and addition new relation with (10), we get

$$2\pounds_V g_{ij} - g_{im}\nabla_j v^m + g_{im}\partial_{\bar{j}}(v^{\bar{m}}) - g_{jm}\nabla_i v^m + g_{jm}\partial_{\bar{i}}(v^{\bar{m}})$$
$$-g_{mi}\{y^b v^c K_{icb}^m - v^{\bar{b}}\Gamma_{bi}^m - \delta_i(v^{\bar{m}})\} - g_{mi}\{y^b v^c K_{jcb}^m - v^{\bar{b}}\Gamma_{bj}^m$$
$$-\delta_j(v^{\bar{m}})\} = 4\bar{\Omega} g_{ij},$$

By attention to (7), (11) and equation $\pounds_V g_{ij} = g_{im}\nabla_j v^m + g_{jm}\nabla_i v^m$, we have $\pounds_V g_{ij} = 2\bar{\Omega} g_{ij}$. This shows the scalar function $\bar{\Omega}$ on TM depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h), thus we can regard $\bar{\Omega}$ as a function on M, and V is an infinitesimal conformal transformation on M. □

In the following we write Ω instead of $\bar{\Omega}$.

Proposition 5. *The vertical components* (v^h) *of* X *can be written as the following form:*

$$v^{\bar{h}} = y^r A_r^h + B^h, \qquad (12)$$

where A_r^h *and* B^h *are the components of a certain* $(1,1)$ *tensor field* A *and a certain contravariant vector field* B *on* M *, respectively*.

Proof. By derivation of (11) respect $\partial_{\bar{r}}$, we get

$$g_{mi}\partial_{\bar{r}}\partial_{\bar{j}}(v^{\bar{m}}) + g_{mj}\partial_{\bar{r}}\partial_{\bar{i}}(v^{\bar{m}}) = 2g_{ij}\partial_{\bar{r}}(\Omega).$$

Since the scalar function Ω on TM depends only on the variables (x^h), thus we have $\partial_{\bar{r}}(\Omega) = 0$. Therefore we get

$$g_{mi}\partial_{\bar{r}}\partial_{\bar{j}}(v^{\bar{m}}) + g_{mj}\partial_{\bar{r}}\partial_{\bar{i}}(v^{\bar{m}}) = 0.$$

Then we have

$$g_{mi}\partial_{\bar{r}}\partial_{\bar{j}}(v^{\bar{m}}) = -g_{mj}\partial_{\bar{r}}\partial_{\bar{i}}(v^{\bar{m}}) = -\partial_{\bar{i}}(g_{mj}\partial_{\bar{r}}(v^{\bar{m}}))$$
$$= -\partial_{\bar{i}}(-g_{mr}\partial_{\bar{j}}(v^{\bar{m}}) + 2\Omega g_{jr})$$
$$= g_{mr}\partial_{\bar{i}}\partial_{\bar{j}}(v^{\bar{m}}) = \partial_{\bar{j}}(g_{mr}\partial_{\bar{i}}(v^{\bar{m}}))$$
$$= \partial_{\bar{j}}(-g_{mi}\partial_{\bar{r}}(v^{\bar{m}}) + 2\Omega g_{ri})$$
$$= -g_{mi}\partial_{\bar{j}}\partial_{\bar{r}}(v^{\bar{m}}) = -g_{mi}\partial_{\bar{r}}\partial_{\bar{j}}(v^{\bar{m}}),$$

which implies that $g_{mi}\partial_{\bar{r}}\partial_{\bar{j}}(v^{\bar{m}}) = 0$. This shows that $\partial_{\bar{j}}(v^{\bar{m}})$ depends only on the variables (x^h). Hence $v^{\bar{h}}$ can be written as $v^{\bar{h}} = y^r A_r^h + B^h$, where A_r^h and B^h are certain function on M. The coordinate transformation rule implies A_r^h and B^h are the components of a certain $(1,1)$ tensor field A and a certain contravariant vector field B. □

Proposition 6. *The vector field* $B = (B^h)$ *is an infinitesimal isometry on* M .

Proof. Substituting equation (12) into the equation (10) and (11), then by Proposition 4, we can get

$$A_{ij} - \nabla_j v_i + \nabla_j B_i = 0, \qquad (13)$$

$$\nabla_j A_i^h + K_{rij}^h v^r = 0, \qquad (14)$$

$$A_{ij} + A_{ji} = 2\Omega g_{ij}. \qquad (15)$$

From equation (13), (15) and Proposition 4, we have

$$\pounds_B g_{ij} = \nabla_j B_i + \nabla_i B_j = 0.$$

Thus the vector field B is an infinitesimal isometry on M. □

Proposition 7. *The scalar function* Ω *on* M *is a constant function.*

Proof. From equation (11), (12), we have

$$2\nabla_k(\Omega g_{ij}) = \nabla_k(A_{ij} + A_{ji})$$
$$= -K_{akji}v^a - K_{akij}v^a = 0.$$

Thus the scalar function Ω on M is constant. □

By proposition 7, the vector field V on M become infinitesimal homothetic transformation.

Conversely, let $V = (v^h)$ be an infinitesimal homothetic transformation on M that is, there exists a constant c such that $\pounds_V g_{ij} = 2cg_{ij}$. Then we define the vector field X on TM as follows

$$X = v^h X_h + y^a \nabla_a v^h X_{\bar{h}}.$$

Proposition 8. *The vector field* X *on* TM *defined above is an infinitesimal conformal transformation*.

Proof. From lemma 3 we have:

$$
\begin{aligned}
£_X G = & -2a(L^2) g_{jm} \{ y^b v^c K_{icb}{}^m - y^a \nabla_a v^b \Gamma_{bi}{}^m \\
& -\delta_i (y^a \nabla_a v^b) \} dx^j \, dx^i \\
& + 2a(L^2) \{ 2\bar{\varphi} g_{ij} + 2c g_{ij} - g_{jm} \nabla_i v^m \\
& + g_{jm} \partial_{\bar{i}} (y^a \nabla_a v^m) \} dx^j \, \delta y^i \\
& + 2a(L^2) g_{mi} \{ y^b v^c K_{jcb}{}^m - y^a \nabla_a v^b \Gamma_{bj}{}^m \\
& -\delta_i (y^a \nabla_a v^b) \} dx^j \, \delta y^i \\
& + 2a(L^2) \{ \bar{\varphi} g_{ij} + g_{jm} \partial_{\bar{i}} (y^a \nabla_a v^m) \} \delta y^j \, \delta y^i \\
= & -2a(L^2) g_{jm} \, y^a \{ v^c K_{icb}{}^m - \nabla_a v^b \Gamma_{bi}{}^m \\
& -\partial_i \nabla_a v^m + \Gamma_{ai}{}^b \nabla_a v^m \} dx^j \, dx^i \\
& + 4a(L^2)(\bar{\varphi} + c) g_{ij} dx^j \, \delta y^i \\
& + 2a(L^2)(\bar{\varphi} g_{ij} + g_{jm} \nabla_i v^m) \delta y^j \, \delta y^i \\
= & \, 4a(L^2)(\bar{\varphi} + c) g_{ij} dx^j \, \delta y^i \\
& + 2a(L^2)(\bar{\varphi} + c) g_{ij} \delta y^j \, \delta y^i \\
= & \, 2(\bar{\varphi} + c) \, G
\end{aligned}
$$

Thus we have $£_X G = 2\bar{\Omega} G$. This shows the vector field X on TM is an infinitesimal conformal transformation. □

Proof of Theorem. Summing up proposition 1 to proposition 5, it is clear that the correspondence $X \to V$ gives a homomorphism of the Lie algebra of infinitesimal fiber-preserving conformal transformations on TM onto the Lie algebra of infinitesimal homothetic transformations on M, and the kernel of this homomorphism is naturally isomorphic onto the Lie algebra of infinitesimal isometries of M.

4. References

[1] M. T. K. Abbassi, Note on the classification theorems of g-natural metrics on the tangent bundle of a Riemannian manifold (M, g), Commnet. Math. Univ. Carolinae, **45** (4) (2004), 591-596.

[2] H. Akbar-Zadeh, Transformations infinitesimals conformes des varietes finsleriennes compactes, Ann. Polon. Math., **36** (1979), 213-229.

[3] M. Anastasiei, Locally conformal Kaehler structures on tangent manifold of a space form, Libertas Math., **19** (1999), 71-76.

[4] I. Hasegawa and K. Yamauchi, Infinitesimal projective transformations on tangent bundles with lift connection, Scientiae Mathematicae Japonicae **52** (2003), 469-483.

[5] R. Miron, and M. Anastasiei,, The Geometry of Lagrange spaces: Theory and applications, Kluwer Acad. Publ, FTPH, no.59,(1994).

[6] R. Miron, and M. Anastasiei, Vector bundles and Lagrange spaces with application to Relativity. Geometry Balkan Press, Romania, (1981).

[7] K. Yamauchi , On infinitesimal conformal transformations of the tangent bundles over Riemannian manifolds, Ann. Rep. Asahikawa. Med. Coll.Vol. 15. 1994.

[8] K. Yano, The theory of Lie Derivatives and Its Applications, North Holland, (1957).

[9] K. Yano and S. Ishihara, Tangent and Cotangent Bundles, Marcel Dekker, New York, (1973).

[10] K. Yano and S. Kobayashi, Prolongations of tensor fields and connection to tangent bundle I, General theory, J. Math. Soc. Japan, **18** (1996), 194 -210.