



## *Partial Eigenvalue Assignment in Discrete-Time Descriptor Systems via Derivative State Feedback*

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### **ABSTRACT**

A method for solving the descriptor discrete-time linear system is focused. For it is easily converted into a standard discrete-time linear system by the definition of a derivative state feedback. Then, a partial eigenvalue assignment is used to have a stable standard system and obtain the state feedback. In a partial eigenvalue assignment, just a part of the open loop spectrum of the standard linear systems is reassigned while leaving the rest of the spectrum invariant, and for reassigning, similarity transformation is used. Using a partial eigenvalue assignment is easier than using eigenvalue assignment. Because by a partial eigenvalue assignment, the size of matrices and state and input vectors are decreased and stability is kept, as well. The eigenvalues of two closed-loop matrices of the descriptor and standard systems are the inverse of each other. Therefore, the stability in PEVA for the descriptor system is kept by reassigning eigenvalue in the unit circle and unchanging the remaining of eigenvalues in the standard system. Also, concluding remarks and an algorithm proposed to the descriptions will be obvious. At the end, the convergence of state and input vectors in the descriptor system to balance point (zero) are shown by figures in a numerical example.

### **KEYWORDS:**

Descriptor Discrete-Time System, Derivative State Feedback, Partial Eigenvalue Assignment, Converge to Balance Point

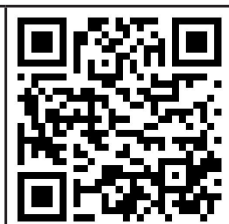
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## 1- INTRODUCTION

Descriptor systems that are also called singular systems are more general and precise than a normal model to depict a dynamical physical. Applications of descriptor systems can be found in various fields such as artificial neuron networks, circuits systems, chemical processes, economics, biologic, power, modeling of mechanical multibody systems, etc. [6,10,11,20,25,28]

Some of the first fundamental works on eigenstructure assignment in descriptor linear systems were established in the 1980s by a number of researchers, such as Cobb (1981) [5], Armentano (1984) [1], Fletcher (1986) [9], Ozcaldiran and Lewis (1987) [21].

In recent years, there are many subjects related to these problems like switched descriptor systems and eigenvalue assignment in the state feedback control for uncertain systems [22,27]. Also, Karbassi et al. worked on non-linear state feedback controllers like in [19].

In the available literature on descriptor systems, there are two kinds of stabilization problems for singular systems. One consists of designing a state or output feedback controller in such a way that the closed-loop system is regular, impulse-free, and stable or equivalently admissible. The other is to design a state or output feedback controller in order to make the closed-loop system regular and stable. Concerning the stability analysis and the stabilization problem, a number of approaches assuming or not assuming the regularity of the descriptor system have been proposed in the literature. Let us quote for instance [2,6,26] among those assuming the regularity and [6,26] without assuming the regularity. Furthermore, positivity and stability of linear descriptor systems have been investigated in [13,15] for systems with regular pencils.

Many practical applications such as the design of large and sparse structures, electrical networks, power systems, computer networks, etc., give rise to very large and sparse problems and the conventional numerical methods for EVA problem do not work well. Furthermore, in most of these applications, only a small number of eigenvalues, which are responsible for instability and other undesirable phenomena, need to be reassigned. Clearly, a complete EVA, in case when only a few eigenvalues are bad, does not make sense. These considerations give rise to the partial eigenvalue assignment (PEVA) problem for the linear

control system such that undesirable eigenvalues are reassigned and other eigenvalues unaltered. An explicit solution to the partial eigenvalue problem by using one of orthogonality relations between eigenvectors for matrix polynomial is considered in [23]. The conditions for the existence and uniqueness of the solution for the single-input problem were given in [24] and for multi-input were presented in [8].

In this paper, a method for finding the solution of descriptor discrete-time linear systems will be investigated. Our method is mixed of PEVA, EVA by similarity transformation and a useful method to convert the descriptor discrete-time linear system into the standard discrete-time linear system. First, the descriptor discrete-time linear system (1) is converted into the standard discrete-time linear system (6) by the definition of the derivative state feedback (2) that is calculated by the PEVA method (section 3). The solution of standard discrete-time linear system (6) or equivalently the original system, i.e., the descriptor discrete-time linear system (1) is obtained by PEVA. On the other hand, we need to reassign undesired eigenvalues of open-loop spectrums in a new system with smaller sizes of matrices such that other eigenvalues unchanged. Also, a theorem for existence and uniqueness of the solution for PEVA in multi-input is represented. Then, feedback in the previous system (before using PEVA) are obtained by an easy relationship between this feedback and gained feedback by PEVA by (17). It is important to say for reassigning undesired eigenvalues, similarity transformation (section 4) is used that is a simple method with a high accuracy.

As mentioned earlier, it is clear that our method has some advantages in which solving the descriptor discrete-time linear systems will be easier. The first advantage is converting descriptor discrete-time linear system into the standard discrete-time linear system because working on standard systems is easier than descriptor systems. EVA has been an applicable method for finding a solution in standard systems and their stability, but by PEVA just by reassigning a part of open-loop matrix spectrum in standard systems while keeping other eigenvalues invariant, their stability are kept. In PEVA we decrease the size of matrices and state and input vectors. It is obvious that calculating is easier than EVA and obtaining state feedback is so comfortable by state feedback governed in PEVA which are other advantages of

our method. Therefore, the state and input vectors in the original system, i.e., descriptor discrete-time linear system, converge into balance point which are displayed by figures in our example. It is worthy to mention that we do not need some assumptions like not having eigenvalues near zero and some criteria on some vectors and being distinct eigenvalues by orthogonality relations for PEVA in [23] or dealing with full row rank matrices in every performed algorithm and finding index of Shuffle and Drazin for descriptor systems in [4,12,14,16] and these are other excellence of the method in this paper. What is more, this method can be used for continuous-time descriptor linear systems by defining a suitable state feedback.

This paper is organized as follows. Next section, presents the convergence of the descriptor discrete-time linear system into the standard discrete-time linear system which both systems have invertible eigenvalues for closed-loop of their matrices. The PEVA problem for obtaining the derivative state feedback is displayed in section 3. Section 4 proposes the similarity transformation to reassign eigenvalues in PEVA. An algorithm and numerical results are presented in sections 5 by an algorithm with all proposed details in its previous sections and a numerical example with the results of all steps of the algorithm in it. Also, the convergence of state and input vectors to balance point, i.e. zero, by their figures are shown. At the final section, concluding remarks are given.

The following notation will be used:  $\mathfrak{R}$ = the set of real numbers,  $\mathfrak{C}$ = the set of complex numbers,  $\mathfrak{R}^{n \times m}$ = the set of  $n \times m$  real matrices and  $\mathfrak{R}^m = \mathfrak{R}^{m \times 1}$ ,  $A^T$ = the transposed matrix of  $A$ ,  $\Omega(A)$ = spectrum of eigenvalues of the matrix  $A$ ,  $I_n$ = the unit matrix of size  $n$ .

## 2- STATEMENT OF THE PROBLEM

Consider the descriptor linear time-invariant controllable system of the form as in the following:

$$Ex_{k+1} = Ax_k + Bu_k ; k \in \mathbb{Z}_+ = \{0,1,2,\dots\} \quad (1)$$

where  $E \in \mathbb{R}^{n \times n}$  with  $\text{rank}(E) \leq n$ ,  $x_k \in \mathbb{R}^n$  is the state vector and  $u_k \in \mathbb{R}^m$  is input vector. It is assumed that  $1 \leq m \leq n$ ,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  are open-loop and input matrices, respectively. Also  $x_0$  is a nonzero definite vector.

Consider the descriptor system (1) with the derivative state feedback:

$$u_k = Fx_{k+1} \quad (2)$$

The aim is the eigenvalue assignment to design a derivative state feedback controller matrix producing a closed-loop system of (1) via feedback (2) with a satisfactory response by shifting  $p(\leq n)$  controllable poles of  $L = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  from undesirable to desirable locations where  $\lambda_i \in \mathfrak{C}$  and  $\lambda_i \neq 0$  and are self-conjugate complex numbers for  $i=1,2,\dots,n$  while other eigenvalues of open-loop matrix remain unchanged, i.e., PEVA.

To establish the proposed results, consider the following assumptions:

- I)  $\text{rank}[E|B]=n$ ,
- II)  $\text{rank}[A]=n$ ,
- III)  $\text{rank}[B]=m$ .

It is clear that if assumption (I) holds, then there exists  $F$  such that [3]:

$$\text{rank}[E - BF] = n \quad (3)$$

For  $F$  such that (3) holds, then from (2) it follows that (1) can be rewritten like a standard linear system, given by:

$$Ex_{k+1} = Ax_k + BFx_{k+1} \Rightarrow x_{k+1} = (E - BF)^{-1} Ax_k \quad (4)$$

**Lemma 2.1:** Consider a matrix  $M \in \mathbb{R}^{n \times n}$ , with  $\text{rank}(M)=n$  and the eigenvalues equal to [18,19]:  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Then, the eigenvalues of  $M^{-1}$  are the following:  $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_n^{-1}$ .

**Remark 2.1:** Consider that  $\lambda = a+bi$  is an eigenvalue of  $M$ , then from Lemma 2.1:

$$\lambda^{-1} = (a + bi)^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

is also an eigenvalue of  $M^{-1}$ .

**Theorem 2.1:** Define the matrices  $N$  and  $M$  as (5) and suppose  $(N, M)$  are controllable:

$$N = A^{-1}E, M = -A^{-1}B \quad (5)$$

Also, let  $F$  be state feedback matrix, such that  $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_n^{-1}\}$  are the eigenvalues of the closed-loop system

$$\begin{cases} z_{k+1} = Nz_k + Mw_k \\ w_k = Fz_k \end{cases} \quad (6)$$

where  $\lambda_i \in \mathfrak{C}$  and  $\lambda_i \neq 0$ ,  $i=1,2,\dots,n$  are arbitrarily assigned. Then for this gained  $F$ , the desired spectrum  $L = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  is the eigenvalues of the controlled system (1) with derivative feedback (2) and also the condition (3) holds.

**Proof:** Considering that  $(N, M)$  are controlled, then we can find a state feedback matrix  $F$  such that the closed-loop of the controlled system with control law (6) given by:

$$z_{k+1} = (N + MF)z_k \quad (7)$$

has poles equal to  $L^{-1}=\{\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_n^{-1}\}$ . Now by (5) note that:

$(N+MF)^{-1}=(A^{-1}(E-BF))^{-1}$ , so  $(N+MF)^{-1}=(E-BF)^{-1}A$  and from (7) and Lemma 2.1, the spectrum  $L=\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  is the eigenvalue of the closed-loop matrix  $(E-BF)^{-1}A$ . Therefore, (3) holds and the eigenvalue of the closed-loop system (1) and feedback (2) are equal to  $L=\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ .

### 3- PARTIAL EIGENVALUE ASSIGNMENT (PEVA)

In this section, we will describe a method to find the feedback  $F$  in system (1) by derivative state feedback (2). At first, some definitions and theorems that we need for the existence and uniqueness theorem for multi-input and single-input PEVA problem are proposed. Next, we bring the Existence and Uniqueness Theorem and its proof and by the description of its proof, the PEVA method is displayed to obtain the derivative feedback  $F$  in system (6).

**Theorem 3.1.** Eigenvector Criterion of controllability [8]. The standard system (6) or, equivalently, the matrix pair  $(N,M)$  is controllable with respect to the eigenvalue  $\lambda$  of  $N$  if  $y^H M \neq 0$  for all  $y \neq 0$  such that  $y^H N = \lambda y^H$ .

**Definition 3.1:** The standard system (6) or the matrix pair  $(N,M)$  is partially controllable with respect to the subset  $\lambda_1, \lambda_2, \dots, \lambda_p$  of the spectrum of  $N$  if it is controllable with respect to each of the eigenvalues  $\lambda_j, j=1, 2, \dots, p$  [8].

**Definition 3.2:** The standard system (6) or the matrix pair  $(N,M)$  is completely controllable if it is controllable with respect to every eigenvalue of  $N$  [8].

**Theorem 3.2:** Existence and Uniqueness for Eigenvalue Assignment Problem [7]. The eigenvalue assignment problem for the pair  $(N,M)$  is solvable for any arbitrary set  $\{\mu_1, \mu_2, \dots, \mu_n\}$  if and only if  $(N,M)$  is completely controllable. The solution is unique if and only if the system is a single-input system (that is if  $M$  is a vector). In the multi-input case, there are infinitely many solutions, whenever a solution exists.

**Theorem 3.3:** Existence and Uniqueness for partial eigenvalue assignment Problem. Consider the pair  $(N,M)$  in system (6) and let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalue of  $N \in \mathbb{C}^{n \times n}$ ,  $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_p, \lambda_{p+1}, \dots, \lambda_n\}$ ,  $A_1 = \text{diag}\{\lambda_1, \dots, \lambda_p\}$  and  $A_2 = \text{diag}\{\lambda_{p+1}, \dots, \lambda_n\}$  be the diagonal matrices containing the given eigenvalues and the sets  $\{\lambda_1, \lambda_2, \dots, \lambda_p\}$  and  $\{\lambda_{p+1}, \lambda_{p+2}, \dots, \lambda_n\}$  be disjointed. Also, let the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_p$  to be changed to  $\mu_1, \mu_2, \dots, \mu_p$  and  $\lambda_{p+1}, \lambda_{p+2}, \dots, \lambda_n$  stay invariant.

Now, if the pair  $(N,M)$  is partially controllable with respect to the set  $\{\lambda_1, \lambda_2, \dots, \lambda_p\}$  then partial eigenvalue assignment problem for the pair  $(N,M)$  solvable for any desired choice of the closed-loop eigenvalues  $\mu_1, \mu_2, \dots, \mu_p$  and vice versa. If the system is a completely controllable single-input system, the solution is unique and there are infinitely many solutions, in the multi-input case and single-input case, when the system is not completely controllable and whenever a solution exists.

**Proof:** We first prove the necessity. Suppose the pair  $(N,M)$  is not controllable with respect to some  $\lambda_j, j=1, 2, \dots, p$ . Then, there exists a vector  $y \neq 0$  such that  $y^H(N-\lambda_j I) = 0$  and  $y^H M = 0$ . This means that for any  $F$ , we have  $y^H(N+MF-\lambda_j I) = 0$ , which implies that  $\lambda_j$  is an eigenvalue of  $N+MF$  for every  $F$ , and thus  $\lambda_j$  cannot be reassigned.

Next, we prove the sufficiency. Considering  $A_1$  and  $A_2$ , we need to prove that there exists a feedback matrix  $F$  which assigns the eigenvalues in  $A_1$  arbitrarily while keeping all the other eigenvalues unaltered.

Let  $X = \{x_1, x_2, \dots, x_n\}$  and  $Y = \{y_1, y_2, \dots, y_n\}$  be, respectively, the right and left eigenvector matrix of  $N$ , and let  $Y_1 = \{y_1, y_2, \dots, y_p\}$ . Since  $Y^H X = I$  and  $Y^H N X = \text{diag}(A_1, A_2)$ , then the partial controllability of the pair  $(N,M)$  with respect to eigenvalues in  $A_1$  implies the partial controllability of the pair  $(\text{diag}(A_1, A_2), Y^H M)$  with respect to the same eigenvalues. Therefore, the pair  $(A_1, Y_1^H M)$  is completely controllable because  $\{\lambda_1, \lambda_2, \dots, \lambda_p\} \cap \{\lambda_{p+1}, \dots, \lambda_n\} = \emptyset$ .

By Theorem 3.2, there exists a feedback matrix  $K$  (see system (16)) such that the closed-loop matrix  $A_1 + Y_1^H M K$  has the desired eigenvalues  $\mu_1, \mu_2, \dots, \mu_p$ .

Denote

$$F = K Y_1^H \tag{8}$$

Then the eigenvalues of a closed-loop matrix are exactly as required. This is seen as follows:

$$\begin{aligned} \{\mu_1, \dots, \mu_p, \lambda_{p+1}, \dots, \lambda_n\} &= \Omega(\text{diag}(\Lambda_1, \Lambda_2)) \\ + Y^H M (K, 0) &= \Omega(Y^H (N + M((K, 0) Y^H)) X) \tag{9} \\ &= \Omega(N + M(K Y_1^H)) \end{aligned}$$

The uniqueness of the solution in the single-input case that is completely controllable and the existence of infinitely many solutions in the multi-input case follow directly from Theorem 3.2.

To complete the proof, we need to show that infinitely many solutions to the PEVA problem are possible when  $M$  is a vector (single-input case) and there exists an uncontrollable eigenvalue  $\lambda_k$  for some  $k > p$  (that is, the associated  $k^{\text{th}}$  right eigenvector  $y_k$  is

such that  $y_k^H N = \lambda_k y_k^H$  and  $y_k^H M = 0$ .

Let  $F$  be a solution to the partial eigenvalue assignment problem. Denote the left and right eigenvectors of the closed-loop matrix  $N_c = N + MF$  by  $Y_c$  and  $X_c$ . Clearly  $y_k^H N_c = y_k^H (N + MF) = \lambda_k y_k^H$  and thus  $y_k$  is also the  $k^{th}$  column of  $Y_c$ . Let  $F\beta = \beta y_k^H$ , where  $\beta$  is an arbitrary scalar. As in (3.9), we can show that the eigenvalues  $\mu_1, \dots, \mu_p, \lambda_{p+1}, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_n$  of  $N_c$  remain unchanged by the application of feedback  $F_\beta$ . Furthermore, the eigenvalue  $\lambda_k$  of  $N_c$  also remains unchanged by the feedback  $F_\beta$ , since the pair  $(N_c, M)$  is not controllable with respect to  $\lambda_k$  by the necessity part of this theorem. Thus,

$$\begin{aligned} \Omega(N + MF) &= \Omega(N_c) = \Omega(N_c + MF_\beta) \\ &= \Omega(N + M(F + \beta y_k^H)) \end{aligned} \tag{10}$$

showing that if  $F$  is a solution, so is  $F + \beta y_k^H$  for an arbitrary  $\beta$ .

Suppose that  $\Omega(N) = \{\lambda_1, \dots, \lambda_p, \lambda_{p+1}, \dots, \lambda_n\}$  which  $p$  is the number of undesired eigenvalues of  $\Omega(N)$  for the pair of  $(N, M)$  in system (6) and assume the set  $S = \{\mu_1, \dots, \mu_p\}$  be closed under complex conjugation which  $rank(M) = rank(B) = m \leq p$ . The aim of PEVA problem is to look for the derivative state feedback  $F$  such that:

$$\Omega(N + MF) = \{\mu_1, \dots, \mu_p, \lambda_{p+1}, \dots, \lambda_n\} \tag{11}$$

and also the sets  $\{\lambda_1, \dots, \lambda_p\}$  and  $\{\lambda_{p+1}, \dots, \lambda_n\}$  be disjointed.

This means that finding  $F$  which reassigns eigenvalues  $\{\lambda_1, \dots, \lambda_p\}$  arbitrarily while keeping all the other eigenvalues,  $\{\lambda_{p+1}, \dots, \lambda_n\}$ , unaltered.

First, we need to obtain left eigenvector of matrix  $N$  as follows:

$$Y = (y_1, y_2, \dots, y_n) \tag{12}$$

Then, we put columns  $y_1, y_2, \dots, y_p$  of  $Y$  in  $Y_1$  that are associated columns by eigenvalues  $\lambda_1, \dots, \lambda_p$ . Therefore,

$$Y = (y_1, y_2, \dots, y_n) = \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1p} \\ y_{21} & y_{22} & \dots & y_{2p} \\ \vdots & \vdots & \dots & \vdots \\ y_{n1} & y_{n2} & \dots & y_{np} \end{bmatrix}_{n \times p} \tag{13}$$

Now consider the pair of  $(A_1, Y_1^H M)$  as follows:

$$Y_1^H M = \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1p} \\ y_{12} & y_{22} & \dots & y_{2p} \\ \vdots & \vdots & \dots & \vdots \\ y_{1p} & y_{2p} & \dots & y_{np} \end{bmatrix}_{p \times n} \times \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix}_{n \times m} \tag{14}$$

$$\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_p) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_p \end{bmatrix}_{p \times p} \tag{15}$$

As a result, the aim is to find feedback  $K$  in the

system

$$\begin{cases} g_{k+1} = \Lambda_1 g_k + (Y_1^H M) v_k \\ v_k = K g_k \end{cases} \tag{16}$$

such that the eigenvalues of the closed-loop of the system (16) be  $\{\mu_1, \dots, \mu_p\}$ . At the end, for finding  $F$  in system (6), we have:

$$F = K Y_1^H \tag{17}$$

#### 4- SIMILARITY TRANSFORMATION OF THE STATE SPACE

In this section, we describe a method for finding feedback  $K$  in system (16).

Consider the system (16) by defining  $A_1 = A_1 \in \mathbb{R}^{p \times p}$  and  $B_1 = Y_1^H M \in \mathbb{R}^{p \times m}$  as follows:

$$\begin{cases} g_{k+1} = A_1 g_k + B_1 r_k \\ r_k = K g_k \end{cases} \tag{18}$$

instead of system (16) and in order to display similarity transformation on it easier.

To obtain the derivative feedback matrix  $K$  in system (16), consider the state transformation

$$g_k = T \tilde{g}_k \tag{19}$$

where  $T$  can be obtained by elementary similarity operations as described in [18,19]. Substituting (19) into the first relationship of (18) yields

$$\tilde{g}_{k+1} = T^{-1} A_1 T \tilde{g}_k + T^{-1} B_1 r_k$$

It is noted that the transformation matrix  $T$  is invertible. In this way,

$$\tilde{A}_1 = T^{-1} A_1 T ; \tilde{B}_1 = T^{-1} B_1 \tag{20}$$

are in a compact canonical form known as vector companion form [18,19]:

$$\tilde{A}_1 = \begin{bmatrix} R_0 & & \\ I_{p-m} & , & 0_{p-m,m} \end{bmatrix}_{p \times p} ; \tilde{B}_1 = \begin{bmatrix} M_0 \\ 0_{p-m,m} \end{bmatrix}_{p \times m} \tag{21}$$

Here  $R_0$  is a  $m \times p$  matrix and  $M_0$  is an  $m \times m$  upper triangular matrix. Note that the Kronecker invariants of the pair  $(A_1, B_1)$  are regular if the difference between any of them is not greater than one. If Kronecker invariants of the pair of  $(A_1, B_1)$  are regular, then  $\tilde{A}_1$  and  $\tilde{B}_1$  are always in the above form [18]. In the case of irregular Kronecker invariants, some rows of  $I_{p-m}$  in  $\tilde{A}_1$  are displaced [19]. (For more details about Kronecker invariants, see [17])

The state feedback matrix which assigns all the eigenvalues to zero for the transformed pair  $(\tilde{A}_1, \tilde{B}_1)$  is then chosen as:

$$\tilde{\Phi} = -M_0^{-1} R_0 \tag{22}$$

which results in the primary state feedback matrix for the pair  $(A_1, B_1)$  defined as:

$$\Phi = \tilde{\Phi} T^{-1} \tag{23}$$

The transformed closed-loop matrix

$$\tilde{\Gamma}_0 = \tilde{A}_1 + \tilde{B}_1 \tilde{\Phi} \tag{24}$$

assumes a compact Jordan form with zero eigenvalues

$$\tilde{\Gamma}_0 = \begin{bmatrix} & 0_{m,p} & \\ I_{p-m} & & 0_{p-m,m} \end{bmatrix}_{p \times p} \tag{25}$$

**Theorem 4.1:** Let  $D$  be a block diagonal matrix in the form

$$D = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_k \end{bmatrix}_{p \times p}$$

where each  $D_j, j=1,2,\dots,k$  is either of the form

$$D_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}$$

(to designate the complex conjugate eigenvalues  $\alpha_j + i\beta_j$ )

or in the case of real eigenvalues

$$D_j = [d_j]$$

If such that diagonal matrix  $D$  with self-conjugate eigenvalue spectrum is added to the transformed closed-loop matrix,  $\tilde{\Gamma}_0$ , then the eigenvalues of the resulting matrix is the eigenvalues in the spectrum.

**Proof:** The primary compact Jordan form in the case of regular Kronecker invariants is in the form (25).

The sum of  $\tilde{\Gamma}_0$  with  $D$  has the form:

$$\begin{aligned} \tilde{H} &= \tilde{\Gamma}_0 + D \\ &= \begin{bmatrix} & 0_{m,p} & \\ I_{p-m} & & 0_{p-m,m} \end{bmatrix}_{p \times p} + \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_k \end{bmatrix}_{p \times p} \\ &= \left. \begin{bmatrix} D_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & D_l & 0 & \cdots & 0 \\ - & - & - & - & - & - & - \\ I_1 & 0 & \cdots & 0 & D_{l+1} & \cdots & 0 \\ \vdots & \ddots & \cdots & 0 & 0 & \ddots & \vdots \\ 0 & \cdots & I_r & 0 & 0 & \cdots & D_k \end{bmatrix}_{p \times p} \right\} \begin{matrix} m \\ p-m \end{matrix} \tag{26} \end{aligned}$$

and also  $I_s, s=1,2,\dots,r$  is the unit matrix of size 2 in case  $p-m$  is even. In case  $p-m$  is odd, only one  $I_s$  takes the form of a unit matrix of size one.

By expanding  $\det(\tilde{H} - \lambda I)$  along the first row, it is obvious that the eigenvalues of  $\tilde{H}$  are the same as those of  $D$ . For the case of irregular Kronecker invariants [19], only some of the unit columns of  $I_{p-m}$  are displaced, since the unit elements are always below the main diagonal, the proof applies in the same manner.

Therefore, the closed-loop system matrix (26) becomes (27). Simple elementary similarity operations can be used to obtain the matrix  $\tilde{H}_\lambda$  from  $\tilde{H}$  such that

$$\tilde{H}_\lambda = \begin{bmatrix} & R_\lambda & \\ I_{p-m} & & 0_{p-m,m} \end{bmatrix} \tag{27}$$

Thus, the primary feedback matrix  $K$  which gives rise to the assignment of eigenvalues  $\{\lambda_1, \dots, \lambda_p\}$  to the system (16) becomes

$$K = \Phi + M_0^{-1} R_\lambda T^{-1} \tag{28}$$

### 5- ALGORITHM AND NUMERICAL EXPERIMENT

In this section, we present an algorithm to obtain the solution of system (1) using partial eigenvalue problem in section 3 based on assigning eigenvalue problem in section 4. Then by an example, we show the simplicity of our method.

**Object:** Assign desired eigenvalues  $\{\mu_1, \dots, \mu_p\}$  to the system (16) and find the matrices  $K$  such that the spectrum of closed-loop system (1), i.e.,  $(E-BF)^{-1}A$  in (3) be  $\{\mu_1, \dots, \mu_p, \lambda_{p+1}, \dots, \lambda_n\}$ .

**Input:** The matrices  $A, B$  and  $E$ .

**Step 1:** Calculate matrices  $N$  and  $M$  from (5).

**Step 2:** Assign desired eigenvalues  $\{\mu_1, \dots, \mu_p\}$  and find  $K$  in system (16).

**Step 2.1:** Obtain  $R_0, M_0$  and  $T^{-1}$  of (19), (20) and (21).

**Step 2.2:** Obtain the state feedback matrix  $\tilde{\Phi}$  and  $\Phi$  by (22) and (23), that assigns zero eigenvalues to pair  $(\tilde{A}_1, \tilde{B}_1)$  and  $(A_1, B_1) = (A_1, Y_1^H M)$  respectively.

**Step 2.3:** Calculate the transformed closed-loop matrix  $\tilde{\Gamma}_0$  by (24) and (25) which assumes a compact Jordan form with zero eigenvalues.

**Step 2.4:** Add a diagonal matrix  $D = \text{diag}(\mu_1, \dots, \mu_p)$  for an arbitrarily set of self-conjugate eigenvalues to  $\tilde{\Gamma}_0$ . Then, the closed-loop system matrix (25) in step 2.3 becomes  $\tilde{H}$  in (26).

**Step 2.6:** Obtain the primary feedback matrix  $K$  that gives rise to the assignment of eigenvalues  $\Omega = \{\mu_1, \dots, \mu_p\}$  to system (16) by  $K = \Phi + M_0^{-1} R_\lambda T^{-1}$  from (28).

**Step 3:** Calculate the derivative state feedback  $F$  in system (16) by (17).

**Step 4:** Find the solution of the system (1) by putting on step 3 in (4).

**Example 5.1:** Consider the descriptor system (1) with following matrices which  $rank(E)=8<9$ .

$$A = \begin{bmatrix} 2 & 6 & 6 & 4 & 3 & -6 & 2 & 6 & -4 \\ 2 & 6 & -4 & -2 & 9 & 0 & -4 & 1 & 4 \\ 6 & 2 & 9 & 1 & 3 & -2 & 9 & 0 & 9 \\ 1 & 0 & 0 & 2 & 2 & 4 & 2 & 0 & -3 \\ 5 & -2 & 4 & 1 & 2 & 5 & 5 & 4 & -5 \\ 4 & 2 & -2 & 2 & 4 & 2 & 9 & 5 & -4 \\ 0 & 6 & -5 & -2 & 2 & -2 & 6 & 0 & 7 \\ 1 & -3 & 9 & 1 & 0 & 2 & 2 & 2 & -3 \\ 1 & 1 & 6 & 1 & 4 & -1 & 1 & 2 & -4 \end{bmatrix}; B = \begin{bmatrix} -5 & 5 & 1 \\ 6 & 2 & 2 \\ -1 & -1 & -4 \\ 9 & 6 & 3 \\ 6 & -4 & -8 \\ 0 & -1 & 6 \\ 2 & -9 & 7 \\ 5 & 2 & 2 \\ 9 & -1 & 1 \end{bmatrix};$$

$$E = \begin{bmatrix} -7 & 4 & -3 & 3 & 3 & -6 & 2 & 6 & -7 \\ 3 & -6 & 0 & -7 & 3 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -9 & 8 & -4 & 2 & 7 & 0 & 2 & 0 \\ -7 & -7 & 4 & 3 & 1 & 2 & 7 & 0 & -3 \\ 3 & 2 & -7 & 2 & -3 & 4 & 2 & 3 & -7 \\ 0 & 3 & 6 & -7 & 3 & -7 & 7 & 6 & 0 \\ 6 & 3 & 7 & 3 & 1 & 0 & 2 & 7 & 7 \\ 2 & -2 & 6 & -4 & 6 & -6 & 2 & 7 & -3 \end{bmatrix}$$

Now we bring the results of all steps in the proposed algorithm.

**Step 1:**

$$N = \begin{bmatrix} -5.21 & -3.16 & -0.45 & 0.1 & 0.65 & -0.7 & 1.03 & -2.77 & -3.13 \\ -4.58 & -3.22 & 4.97 & -2.02 & 2.62 & -2.12 & 2.94 & 0.74 & -0.96 \\ 0.51 & -0.51 & 2.12 & -0.48 & 0.96 & -0.89 & 0.83 & 0.87 & 0.54 \\ 2.96 & 1.19 & -3.03 & 1.47 & -2.19 & 4.08 & -3.25 & -0.55 & 1.16 \\ 3.82 & 1.15 & -1.7 & -0.31 & -0.63 & 0.96 & -1.74 & 0.7 & 0.71 \\ -0.27 & -1.41 & 3.01 & 0.53 & 0.5 & 0.92 & 1.09 & 0.78 & 1.85 \\ 0.75 & 0.52 & 0.36 & -0.44 & 0.14 & -0.45 & 0.32 & 0.75 & -0.52 \\ 2.98 & 3.81 & -2.82 & 2.51 & -2.11 & 1.28 & -0.64 & 0.9 & 2.74 \\ 2.59 & 1.97 & -1.71 & 1.12 & -1.56 & 1.72 & -1.31 & 0.06 & 2.32 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.24 & 1.81 & 5.02 \\ -2.49 & 1.57 & 2 \\ -1.2 & 0.33 & 0.42 \\ 2.31 & -4.97 & -2.74 \\ -0.38 & -1.27 & -2.4 \\ -1.7 & -0.61 & -0.36 \\ -0.52 & 0.86 & -0.9 \\ 2.73 & -1.5 & -2.53 \\ 1.73 & -1.8 & -1.84 \end{bmatrix}$$

**Step 2:**

$$\Omega(N) = \{-6.66, -0.51 \pm 1.92i, 0, 0.24 \pm 0.47i, 0.72, 1.86, 3.61\}$$

Therefore,  $p=4$  and by considering

$$\Lambda_1 = \begin{bmatrix} 0.72 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.24+0.47i & 0 \\ 0 & 0 & 0 & 0.24-0.47i \end{bmatrix}$$

$$Y_1^H M = \begin{bmatrix} 0.28 & 0.03 & 0.24 \\ 0.05 & 0.05 & 0.23 \\ -0.32i & 0.2+0.5i & 0.33+0.19i \\ 0.32i & 0.2-0.5i & 0.33-0.19i \end{bmatrix}$$

the matrix  $\Phi$  which assigns zero eigenvalue to the system (16) is obtained as:

$$\Phi = \begin{bmatrix} -3.77 & 0 & 0.18+1.12i & 0.18-1.12i \\ -3.15 & 0 & -0.33+1.36i & -0.33-1.36i \\ 1.75 & 0 & -0.76i & 0.76i \end{bmatrix}$$

and by reassigning  $\{\pm 10, \pm 5\}$  instead of  $\{0, 0.24+0.47i, 0.72\}$ , the matrix feedback  $K$  is obtained as:

$$K = 10^2 \times \begin{bmatrix} 0.03 & -7.29 & 2.85+0.65i & 2.85-0.65i \\ 0.18 & -0.09 & -0.1-0.07i & -0.1+0.07i \\ 0.13 & -0.21 & -0.06-0.04i & -0.06+0.04i \end{bmatrix}$$

**Step 3:** The derivative state feedback matrix  $F$  for the system (6) is calculated by:

$$F = KY_1^H = 10^2 \times \begin{bmatrix} -1.32 & -1.03 & -0.17 & -0.1 & -1.25 & -0.04 & -1.73 & -0.26 & -1.72 \\ -0.06 & 0 & -0.14 & -0.06 & -0.02 & 0.02 & -0.13 & -0.04 & 0.04 \\ -0.09 & -0.01 & -0.17 & -0.05 & -0.04 & 0.02 & -0.16 & -0.03 & -0.03 \end{bmatrix}$$

Now, we have:

$$\Omega((E - BF)^{-1}A) = \Omega((N + MF)^{-1}) = \{-0.15, -0.12 \pm 0.48i, 0.27, 0.53, \pm 0.1, \pm 0.2\}$$

Figs. 1 and 2 show simulation results when

$$x_0 = [0.001 \quad -0.001 \quad 0.001 \quad -0.001 \quad 0.001 \quad -0.001 \quad 0.001 \quad -0.001 \quad 0.001]^T$$

## 6- CONCLUDING REMARKS

A method for finding the solution of descriptor discrete-time linear system in the form of (1) has been considered. First, by the use of the derivative state feedback (2), a displayed system is converted into a standard discrete-time linear system (6) and it explains the advantages of this method because working with the standard systems is much easier than the descriptor mode. Secondly, the PEVA method standard system based on similarity transformation and assigning zero eigenvalues to our standard system has been used to

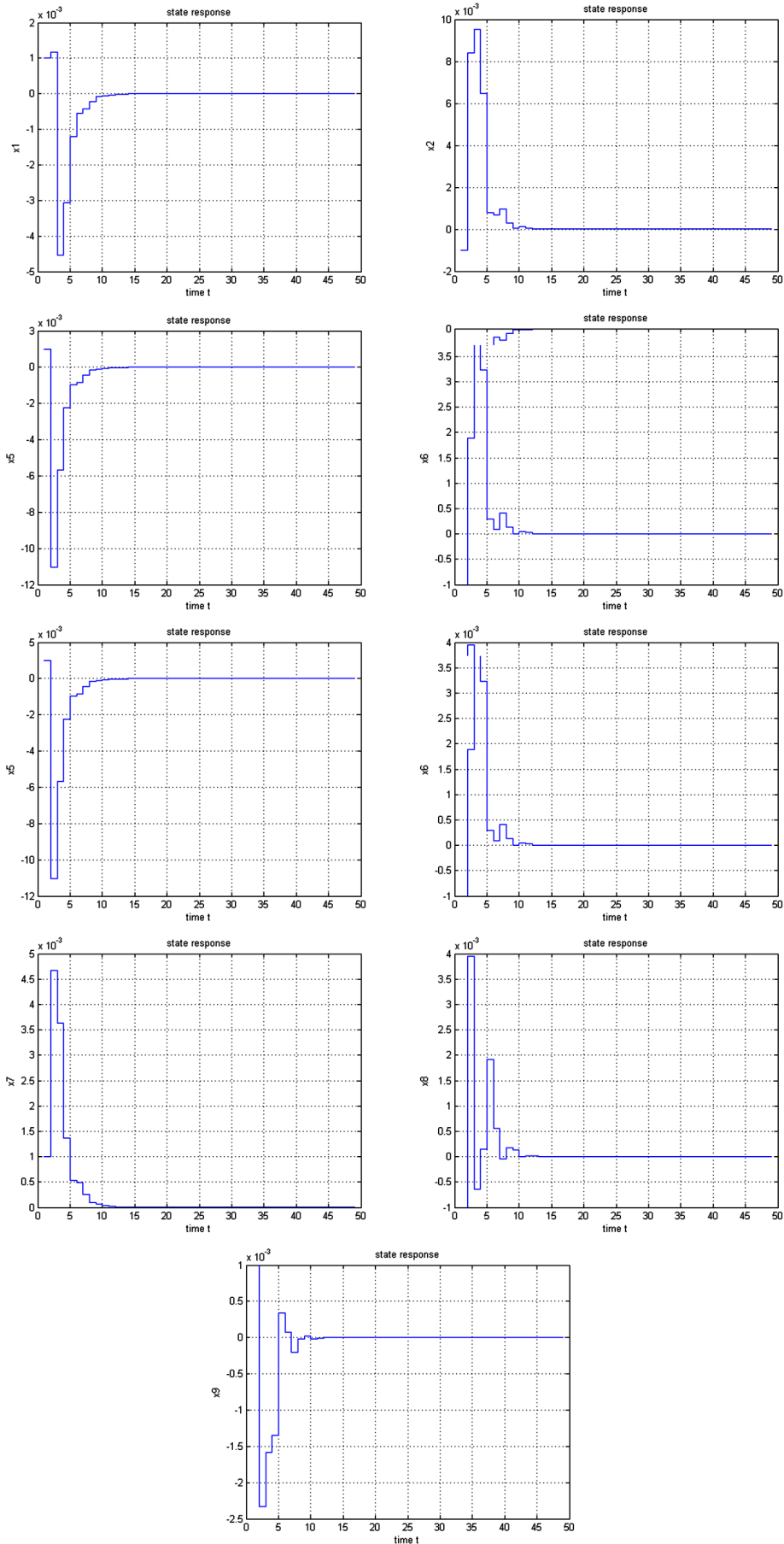


Fig. 1. State vector converging into zero in example 5.1



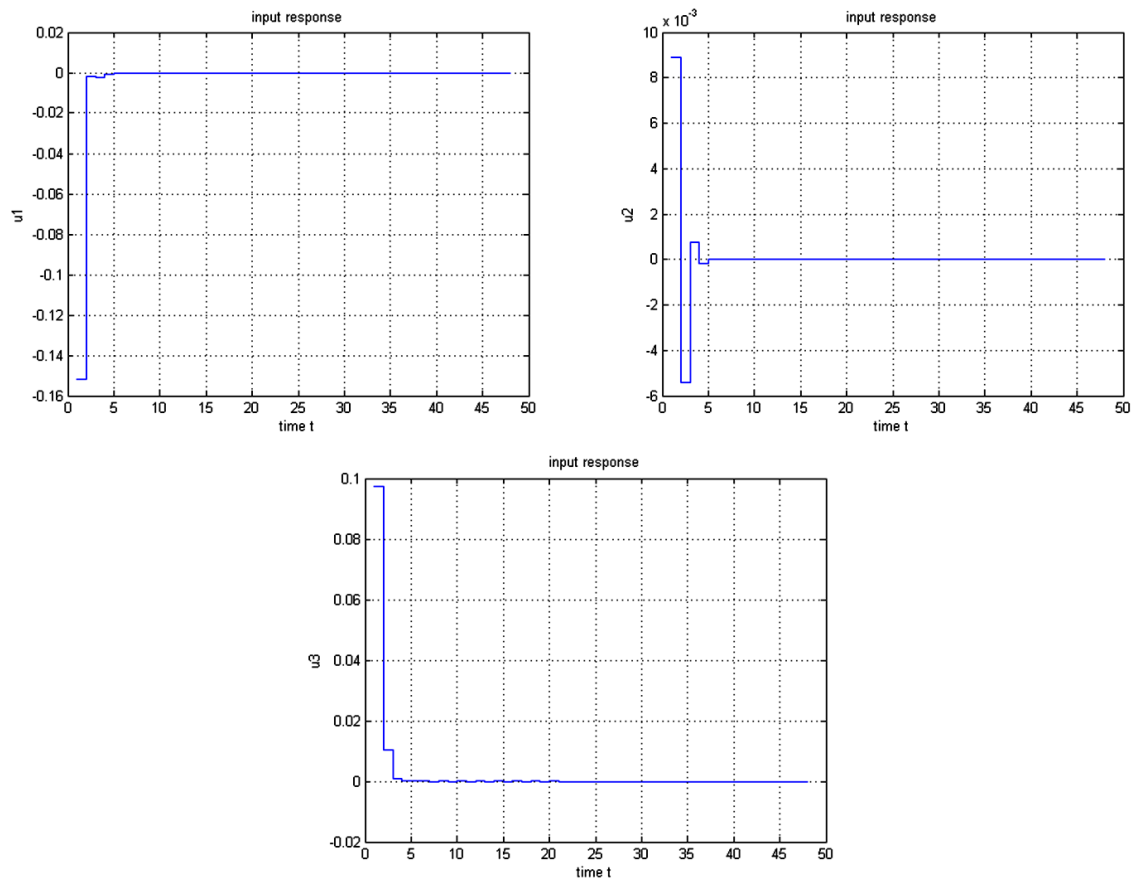


Fig. 2. Input vector converging into zero in example 5.1

obtain  $F$  in (6). Thirdly, the state and input vectors in (1) has been obtained and illustrated by a numerical example which showed the input and state vectors convergence to balance point (zero) which is another advantage of our method.

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