Hybrid of Rationalized Haar Functions Method for Mixed Hammerstein Integral Equations

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ABSTRACT

A numerical method for solving nonlinear mixed Hammerstein integral equations is presented in this paper. The method is based upon hybrid of rationalized Haar functions approximations. The properties of hybrid functions which are the combinations of block-pulse functions and rationalized Haar functions are first presented. The Newton-Cotes nodes and Newton-Cotes integration method are then utilized to reduce the nonlinear mixed Hammerstein integral equations to the solutions algebraic equations. The method is computationally attractive, and applications are demonstrated through illustrative examples.

KEYWORDS

Hybrid, Rationalized Haar functions, Block-pulse functions, Newton-Cotes, Nonlinear, Mixed Hammerstein integral equation

1. INTRODUCTION

In this paper, I present a hybrid of rationalized Haar functions method for solving nonlinear mixed Hammerstein integral equations. Several numerical methods for approximating the solution of Hammerstein integral equations are known. For Fredholm-Hammerstein integral equations, the classical method of successive approximations was introduced in [1]. A variation of the Nystrom method was presented in [2]. A collocation type method was developed in [3]. In [4], Brunner applied a collocation-type method to nonlinear Volterra-Hammerstein integral equations and integro-differential equations, and discussed its connection with the iterated collocation method. Guoqiang [5] introduced and discussed the asymptotic error expansion of a collocation-type method for Volterra- Hammerstein integral equations.

Orthogonal functions, often used to represent an arbitrary time function have received considerable in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem. Orthogonal functions have also been proposed to solve linear integral equations. Special attention has been given to applications of Walsh functions [6], block-pulse functions [7], Laguerre series [8], Legendre polynomials [9], Chebyshev polynomials [10] and Fourier series [11]. The orthogonal set of Haar functions is a group of square waves with magnitude of $+2^{i/2}, -2^{i/2}$ and 0, $i = 0,1,2,\ldots$ [12]. The use of the Haar functions comes from the rapid convergence feature of Haar series in expansion of function compared with that of Walsh series [13]. Lynch et al. [14] have rationalized the Haar transform by deleting the irrational numbers and introducing the integral powers of two. This modification results in what is called the rationalized Haar (RH) transform. The RH transform preserves all the properties of the original Haar transform and can be efficiently implemented using digital pipeline architecture [15]. The RH functions are composed of only three amplitudes $+1, -1$ and 0. Further, Ohkita and Kobayashi [16]-[17] applied RH functions to solve linear ordinary differential equation [16] and linear first and second order partial differential equations [17].

M. Razzagh and H. Marzban [18] introduced the hybrid of block-pulse and orthogonal polynomials for approximation different problems. In using orthogonal RH functions to get good accuracy the number of variable be very large as $k = 2^{\alpha+1}, \alpha = 0,1,2,\ldots$ so for decrease variable and time and higher accuracy we use idea of hybrid of block-pulse and rationalized Haar (HRH) functions to approximate solution of nonlinear mixed Hammerstein integral equations.

Very few references have been found in the technical literature dealing with Volterra-Fredholm integral equations. Yalcinbas [19] applied Taylor series to the following nonlinear Volterra-Fredholm integral equation

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$$y(t)= f(t)+\lambda_1\int_0^t \kappa_1(t,s)[y(s)]^p\, ds + \lambda_2\int_0^1 \kappa_2(t,s)[y(s)]^q\, ds, \quad 0\le t,s\le 1,$$

where p and q are nonnegative integers and λ_1 and λ_2 are constants. Moreover, $f(t)$, the kernels $\kappa_1(t,s)$ and $\kappa_2(t,s)$ are assumed to have nth derivatives on the interval $0\le t,s\le 1$.

In the present article, we are concerned with the application HRH functions to the numerical solution of a nonlinear mixed Hammerstein integral equation of the form

$$y(t)= f(t)+\lambda_1\int_0^t \kappa_1(t,s)g_1(s,y(s))ds + \lambda_2\int_0^1 \kappa_2(t,s)g_2(s,y(s))ds,\ 0\le t,s\le 1, \tag{1}$$

where $f(t)$, the kernels $\kappa_1(t,s)$ and $\kappa_2(t,s)$ are assumed to be in $L^2(\mathbb{R})$ on the interval $0\le t,s\le 1$. We assume that (1) has a unique solution $y(t)$ to be determined. The method consists of expanding the solution by HRH functions with unknown coefficients. The properties of the HRH functions together with the Newton-Cotes nodes and Newton-Cotes integration [20] are then utilized to evaluate the unknown coefficients and find an approximate solution to (1). In this method time and computations are smaller and more accuracy than [21].

The article is organized as follows: In Section 2, we describe the basic formulation of the HRH functions required for our subsequent development. Section 3 is devoted to the solution of (1) by using HRH functions. In Section 4, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2. Properties of Hybrid Functions

2.1 Hybrid Functions

The HRH functions $\phi_{nr}(t)$, $n=1,2,\dots,N$, $r=0,1,\dots,k-1$, $k=2^{\alpha+1}$, $\alpha=0,1,2,\dots$ defined on [0,1), have three arguments, r and n are the order for RH functions and block-pulse functions, respectively and t, the normalized time, is defined as

$$\phi_{nr}(t)=\begin{cases}\phi_r(Nt+1-n), & \dfrac{n-1}{N}\le t\le \dfrac{n}{N}\\ 0, & otherwise,\end{cases}$$

Here, $\phi_r(t)=RH(r,t)$ are the well-known RH functions of order r, which are orthogonal in the interval [0,1) and satisfy the following formula[23]:

$$RH(r,t)=\begin{cases}1, & J_1\le t< J_{1/2}\\ -1, & J_{1/2}\le t< J_0\\ 0, & otherwise\end{cases}$$

where

$$J_u=\frac{j-u}{2^i}, \quad u=0,\frac{1}{2},1.$$

The value of r is defined by two parameters i and j as

$$r=2^i+j-1,\ \ i=0,1,2,3,\dots\ \ j=1,2,3,\dots,2^i$$

RH(0,t) is defined for $i=j=0$ and is given by

$$RH(0,t)=1, \quad 0\le t<1.$$

Since $\phi_{nr}(t)$ is the combination of RH functions and block-pulse functions which are both complete and orthogonal, thus the set of hybrid functions is complete and orthogonal.

The orthogonality property is given by

$$\int_0^1 \phi_{nr}(t)\phi_{n'r'}(t)dt=\begin{cases}\dfrac{2^{-i}}{N}, & n=n', r=r'\\ 0, & otherwise\end{cases}$$

where

$$r=2^i+j+1, \quad r'=2^{i'}+j'+1.$$

2.2 Function Approximation

A function $f(t)$ defined over $[0,1)$ may be expanded in HRH functions as

$$f(t)=\sum_{r=0}^{\infty}\sum_{n=1}^{\infty}a_{nr}\phi_{nr}(t), \tag{2}$$

where a_{nr} are given by

$$a_{nr}=\frac{(f,\phi_{nr})}{\|\phi_{nr}\|^2}=2^i N\int_0^1 f(t)\phi_{nr}(t)dt,$$
$$n=1,2,3,\cdots,\ r=0,1,2,\cdots$$

and $(.,.)$ denotes the inner product. If the infinite series in (2) is truncated, then (2) can be written as

$$f(t)\simeq\sum_{r=0}^{k-1}\sum_{n=1}^{N}a_{nr}\phi_{nr}(t)=A^T B(t).$$

The HRH functions coefficient vector A and HRH functions vector $B(t)$ are defined as

$$A=[a_{10},a_{11},\dots,a_{1k-1}\,|\,a_{20},a_{21},\dots,a_{2k-1}\,|\, \dots|\,a_{N0},a_{N1},\dots,a_{Nk-1}]^T, \tag{3}$$

and

$$B(t)=[\phi_{10}(t),\phi_{11}(t),\ldots,\phi_{1k-1}(t)\,|\,\phi_{20}(t),\phi_{21}(t),\ldots,\phi_{2k-1}(t)\,|\,\ldots\,|\,\phi_{N0}(t),\phi_{N1}(t),\ldots,\phi_{Nk-1}(t)]^T. \quad (4)$$

Also, the integration of the cross product of two hybrid vectors is

$$\int_0^1 B(t)B^T(t)dt=W=\frac{1}{N}diag(D,D,\ldots,D),$$

where W is $Nk\times Nk$ matrix and

$$D=\frac{1}{N}diag(1,1,\frac{1}{2},\frac{1}{2},\underbrace{\frac{1}{2^2},\ldots,\frac{1}{2^2}}_{2^2},\ldots,\underbrace{\frac{1}{2^\alpha},\frac{1}{2^\alpha},\ldots,\frac{1}{2^\alpha}}_{2^\alpha}).$$

3. Nonlinear Mixed Hammerstein Integral Equations

Consider the nonlinear mixed Hammerstein integral equations given in (1). In order to use HRH functions, we first approximate $y(t)$ as

$$y(t)=A^TB(t), \quad (5)$$

where A and $B(t)$ are defined similarly in (3) and (4). Then from (1) and (5) we have

$$A^TB(t)=f(t)+\lambda_1\int_0^t\kappa_1(t,s)g_1(s,A^TB(s))ds+\lambda_2\int_0^1\kappa_2(t,s)g_2(s,A^TB(s))ds, \quad (6)$$

we now collocate (6) at Nk points t_p as

$$A^TB(t_p)=f(t_p)+\lambda_1\int_0^{t_p}\kappa_1(t_p,s)g_1(s,A^TB(s))ds+\lambda_2\int_0^1\kappa_2(t_p,s)g_2(s,A^TB(s))ds. \quad (7)$$

For a suitable collocation points, we choose Newton-Cotes nodes [20] as

$$t_p=\frac{2p-1}{2Nk},\qquad p=1,2,3,\ldots,Nk.$$

In order to use the Newton-Cotes integration formula for (7), we transfer the Nk intervals $[0,t_p]$ into interval $[0,1]$ by means of the transformation

$$s=t_p\tau,\quad \tau\in[0,1).$$

Let

$$\zeta_1(t_p,s)=\kappa_1(t_p,s)g_1(s,A^TB(s)),$$
$$\zeta_2(t_p,s)=\kappa_2(t_p,s)g_2(s,A^TB(s)).$$

Equation (7) may then be rewritten as

$$A^TB(t_p)=f(t_p)+\lambda_1t_p\int_0^1\zeta_1(t_p,t_p\tau)d\tau+\lambda_2\int_0^1\zeta_2(t_p,\tau)d\tau.$$

By using the Newton-Cotes integration formula, we get

$$A^TB(t_p)=f(t_p)+\lambda_1t_p\sum_{j=1}^{k_1}\omega_{1j}\zeta_1(t_p,t_p\tau_{1j})+\lambda_2\sum_{j=1}^{k_2}\omega_{2j}\zeta_2(t_p,\tau_{2j}),\quad p=1,2,\ldots,Nk, \quad (8)$$

where τ_{1j} and τ_{2j} are k_1 and k_2 Newton-Cotes Nodes in interval $[0,1)$ and ω_{1j}, ω_{2j} are the corresponding weights given in [20]. Equation (8) gives Nk nonlinear equations which can be solved for the elements of A in (5) using Newton's iterative method [20]. Ultimately, the continuous approximate solution $y(t)$ becomes

$$y(t)\simeq f(t)+\lambda_1\int_0^t\kappa_1(t,s)g_1(s,A^T\phi(s))ds+\lambda_2\int_0^1\kappa_2(t,s)g_2(s,A^T\phi(s))ds.$$

4. Illustrative Examples

Example 1.

Consider the nonlinear Volterra-Hammerstein integral equation given in [21] as

$$y(t)=f(t)+\int_0^t\kappa_1(t,s)y^2(s)ds,\ \ 0\le t<1 \quad (9)$$

where $\kappa_1(t,s)=-3\sin(t-s)$, $\lambda_1=1,\lambda_2=0$ and

$$f(t)=1+\sin^2(t).$$

By using the method in section 3 and using (9) is solved. The computational result for $k=4, N=4$ together with the exact solution $y(t)=\cos(t)$ and those of method of [21] are given in Table 1.

TABLE 1.
APPROXIMATES AND EXACT VALUES OF $y(t)$

t	Exact	Method of [21] for k=16	Present Method for k=4, N=4
0	1	1	1
0.2	0.980067	0.9800	0.980066
0.4	0.921061	0.9210	0.921059
0.6	0.825336	0.8255	0.825334
0.8	0.696707	0.6969	0.696705
1	0.540302	0.5405	0.540301

Example 2.

Consider the nonlinear Fredholm integral equation considered in [22]

$$y(t) = t - \frac{\pi}{8} + \frac{1}{2}\int_0^1 \frac{1}{1+y^2(s)}ds, \tag{10}$$

where $\lambda_1 = 0$. We solve (10) using the method in section 3. The computational result for $k = 4, N = 2$ and $k = 4, N = 4,$ together with the exact solution $y(t) = t$ are given in Table 2.

TABLE 2.
APPROXIMATES AND EXACT VALUES OF $y(t)$

t	Exact	Approximate for k=4, N=2	Approximate for k=4, N=4
0	0	0.000030	0.000003
0.2	0.2	0.200025	0.200003
0.4	0.4	0.400037	0.400001
0.6	0.6	0.600051	0.600004
0.8	0.8	0.800046	0.800004
1	1	1.000030	1.000002

Example 3.

In this example HRH functions approximation is used to solve the integral equation reformulation of the nonlinear two-point boundary value problem

$$\frac{d^2y(t)}{dt^2} - e^{y(t)} = 0, \ \ t \in [0,1]; \quad y(0) = y(1) = 0, \tag{11}$$

which is of great interest in hydrodynamics [24]. This problem has a unique solution given in [8] as

$$y(t) = -\ln(2) + \ln(\lambda(t)),$$

where

$$\lambda(t) = \frac{c}{\cos(\frac{1}{2}c(t-\frac{1}{2}))}.$$

Here, c is the root of $[\frac{c}{\cos(c/4)}]^2 = 2$. Problem (11) can be reformulated as the integral equation

$$y(t) = \int_0^1 \kappa(t,s)e^{y(s)}ds, \tag{12}$$

where

$$\kappa(t,s) = \begin{cases} -s(1-t), & s \le t \\ -t(1-s), & t \le s. \end{cases}$$

The proposed method was applied to approximate the solution of (12) with $k = 4, N = 2,$ $k = 4, N = 4$ and $k = 4, N = 6$. In [8], the above problem was solved with the collocation points chosen to be $t_i = \frac{i-1}{N-1}$, $i = 1, 2, \ldots, N$ and the basis functions as piecewise-linear functions, in which a rather large system of nonlinear equations have to be solved to obtain accuracy of comparable order. Table (3) represents the error estimates using the method of [3] together with the results obtained for maximum errors by the proposed method.

TABLE 3.
ERROR ESTIMATES FOR EXAMPLE 3.

Methods	$\|y - \hat{y}\|$
Method of [3]	
N= 5	7.81×10^{-3}
N= 17	5.61×10^{-4}
N= 65	3.66×10^{-5}
Present Method	
k=4, N=2	$< 0.47 \times 10^{-5}$
k=4, N=4	$< 0.52 \times 10^{-6}$
k=4, N=6	$< 0.64 \times 10^{-7}$

Example 4.

Consider the nonlinear mixed Hammerstein integral equation given in [19] by

$$\begin{aligned} y(t) &= \frac{-1}{30}t^6 + \frac{1}{3}t^4 - t^2 + \frac{5}{3}t - \frac{5}{4} \\ &+ \int_0^t (t-s)[y(s)]^2 ds \\ &+ \int_0^1 (t+s)[y(s)]ds, \qquad 0 \le t, s \le 1. \end{aligned} \tag{13}$$

We applied the method presented in this paper and solved (13) with $k = 4, N = 2$ and $k = 4, N = 4$. The computational result together with the exact solution $y(t)=t^2-2$ are given in Table 4.

TABLE 4.
APPROXIMATES AND EXACT VALUES OF $y(t)$

t	Exact	Approximate for k=4, N=2	Approximate for k=4, N=4
0	-2	-2.000034	-2.000002
0.2	-1.96	-1.960032	-1.960005
0.4	-1.84	-1.840025	-1.840005
0.6	-1.64	-1.640034	-1.640002
0.8	-1.36	-1.360029	-1.360003
1	-1	-1.000024	-1.000002

5. CONCLUSION

In the present work, the HRH functions are used to find the solution of nonlinear mixed Hammerstein integral equations. The HRH functions together Newton-Cotes nodes t_p and Newton-Cotes integration formula solution of (1) was converted in a problem of solving a system of algebraic equations. In this method time and computations are smaller and more accuracy than [21]. (see example 1) Illustrative examples are given to demonstrate the validity and applicability of the proposed method.

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