

Evaluation of the Centre Manifold Method for Limit Cycle Calculations of a Nonlinear Structural Wing

Morteza Dardelⁱ, Firooz Bakhtiari-Nejadⁱⁱ

ABSTRACT

In this study the centre manifold is applied for reduction and limit cycle calculation of a highly nonlinear structural aeroelastic wing. The limit cycle is arisen from structural nonlinearity due to the large deflection of the wing. Results obtained by different orders of centre manifolds are compared with those obtained by time marching method (fourth-order Runge-Kutta method). These comparisons show zero, third and fifth order manifolds are very good approximation of this system. The aeroelastic model is a low aspect ratio rectangular cantilevered wing in a low subsonic flow which is structurally modeled by the Von Karman plate theory. A continuous time reduced order modified vortex lattice aerodynamics model is utilized in aerodynamics modeling.

KEYWORDS

Centre manifold, Limit cycle, Nonlinear aeroelasticity, Von Karman plate, Domain of Attraction

1. INTRODUCTION

Aeroelastic systems represent different types of nonlinear behaviors such as chaotic, bifurcation, resonance and limit cycle. These nonlinear behaviors are due to the high nonlinearities present in aerodynamics or structural models. A good review of nonlinear aeroelasticity field and related topics is given in [1]. Among these different types of nonlinearities, special attention is given to the structural nonlinearity. An aeroelastic system with this type of nonlinearity results a pair of complex conjugate eigenvalues. The centre manifold method can be used for determining limit cycle amplitude and stability analysis of such systems.

Many researchers used centre manifold for determining stability, and predicting amplitude and frequency of limit cycles. Liu *et al.* investigated application of the centre manifold theory in predicting limit cycle amplitude and frequency of a two degrees of freedom airfoil with cubic nonlinearities in pitching and plunging motions [2]. A comparison between harmonic balance approach, centre manifold and the time response solution of the Runge-Kutta method is given by Shahrzad and Mahzoon for predicting limit cycle of airfoils with nonlinear pitching stiffness [32]. Dessi *et al.* investigate the limit cycle stability reversal of a two and three degrees of freedom airfoils with nonlinear cubic pitching moment [4,5]. They used singular perturbation technique based on the normal

form method in their analysis. Grzedzinski, investigated limitations in the application of the centre manifold method for limit cycle calculations of a three-dimensional thin airfoil with cubic structural force of the aileron in an incompressible flow [6]. He showed, the centre manifold based on polynomial approximations cannot accurately predict limit cycle amplitude of the aeroelastic system for different types of nonlinearities. Also Qian and Li [7], investigated the flutter of a two dimensional airfoil in a supersonic flow with cubic structural and aerodynamic nonlinearities. They used a normal form algorithm, which combines the normal form theory and the centre manifold together. Their algorithm is a symbolic procedure developed by Bi *et al* [8].

These studies are confined to the structural nonlinearity of airfoils with two or three degrees of freedom. Here, a wing with distributed nonlinearity is studied by using the centre manifold. The wing has in-plane and out-of-plane displacements. The in-plane displacements are due to the stretching of the mid-plane and have double bending in both chord-wise and span-wise directions (Von Karman plate theory). The aerodynamics flow is a low subsonic, incompressible and irrotational and is modeled by the vortex lattice theory [9]. A reduced order aerodynamics model based on the continuous modified vortex lattice model is used in the aerodynamics calculations. Different orders of centre manifold are incorporated to obtain the accurate amplitude prediction of limit cycles. Results show zero,

ⁱ M. Dardel is with the Department of Mechanical Engineering, Amirkabir University of Technology, Tehran, Iran (e-mail: mdardel@gmail.com).

ⁱⁱ Corresponding Author, F. Bakhtiari-Nejad is with the Department of Mechanical Engineering, Amirkabir University of Technology, Tehran, Iran (e-mail: bakhtiari@aut.ac.ir).

third and fifth order manifolds are good approximation of the original system.

2. AEROELASTIC WING MODEL

A schematic diagram of the wing model with a vortex lattice model of the unsteady flow is shown in Fig.1. The rectangular wing has span L , chord c , and thickness of h . The aeroelastic state space equations are derived from Lagrange's equations based on the Von-Karman plate theory [12] using the total kinetic and elastic energies and the work done by applied aerodynamics force.

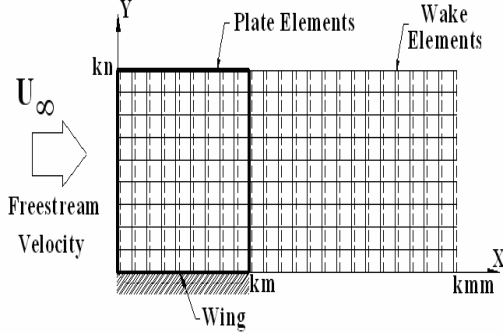


Figure 1: Aeroelastic model.

Approximate modes are substituted into the energy expressions and then into Lagrange's equations to yield equations of motion for each structural modal coordinate (Rayleigh – Ritz method).

For a plate, the stretching energy U_S and bending energy U_B are as follows:

$$U_S = \frac{1}{2} \frac{Eh}{1-\nu^2} \iint \left[\left(\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right)^2 + \left(\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right)^2 + 2\nu \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \right] dx dy \quad (1)$$

$$U_B = \frac{1}{2} \frac{Eh^3}{12(1-\nu^2)} \iint \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\nu) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dx dy \quad (2)$$

where of u , v are in-plane and w is transverse displacements of plate. E , h and ν are Young module, thickness and Poisson's ratio of plate, respectively.

With neglecting the in-plane inertia terms due to their small values, the kinetic energy is given by:

$$T = \frac{1}{2} \iint_A m \dot{w}^2 dx dy \quad (3)$$

The in-plane and transverse displacements can be expressed as follows:

$$u = \sum_i \sum_j a_{ij}(t) u_i(x) u_j(y) \quad (4)$$

$$v = \sum_r \sum_s b_{rs}(t) v_r(x) v_s(y) \quad (5)$$

$$w = \sum_m \sum_n q_{mn}(t) w_m(x) w_n(y) \quad (6)$$

where a_{ij} and b_{rs} and q_{mn} are generalized coordinates, and u_i , u_j , v_r , v_s , w_m , and w_n are appropriate mode functions which satisfy geometric boundary conditions.

Therefore, the Lagrange's equations are as follows [13]:

$$\frac{\partial L}{\partial a_{ij}} = 0 \quad (7)$$

$$\frac{\partial L}{\partial b_{rs}} = 0 \quad (8)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_{mn}} \right) - \frac{\partial L}{\partial q_{mn}} + Q_{Aero} = 0 \quad (9)$$

where $L=T-U$ and Q_{Aero} is the aerodynamics force.

$$Q_{Aero} = \iint \Delta p \frac{\partial w}{\partial q_{mn}} dx dy \quad (10)$$

With substituting energies into the Lagrange's equation, the aeroelastic equations of this model are as follows [10]:

$$\sum_k \sum_p C_{kp}^{ij} \bar{a}_{kp} + \sum_g \sum_f C_{gf}^{ij} \bar{b}_{gf} = C^{ij} \quad (11)$$

$$\sum_k \sum_p D_{kp}^{ij} \bar{a}_{kp} + \sum_g \sum_f D_{gf}^{ij} \bar{b}_{gf} = D^{rs} \quad (12)$$

$$\sum_m \sum_n [A_{mn}^{ij} \ddot{q}_{mn} + B_{mn}^{ij} \bar{q}_{mn}] + F_N^{ij} + Q_{Aero}^{ij} = 0 \quad (13)$$

where, C_{kp}^{ij} and D_{gf}^{rs} are stretch stiffness matrices of wing in span-wise and chord-wise directions, and C_{gf}^{ij} and D_{kp}^{rs} are coupling stretch stiffness between displacements in span-wise and chord-wise directions, respectively. Also C^{ij} and D^{rs} are nonlinear forces which are quadratic polynomials of the plate transverse deflection. A_{mn}^{ij} , B_{mn}^{ij} and F_N^{ij} are mass, stiffness and nonlinear bending stiffness matrices, respectively. F_N^{ij} is

a polynomial of order two and three in terms of inplane and transverse generalized coordinates.

2.1 Discrete Aerodynamics Model

An unsteady vortex lattice method is used for modeling flow and calculating the aerodynamic forces [9]. The flow is assumed incompressible, inviscid, and irrotational. As shown in Fig.1, vortex ring panels are used in aerodynamics modeling. The wing is modeled with $K_m \times K_n$ vortex ring panels, with K_m and K_n panels in stream-wise and span-wise directions, respectively. For aerodynamics modeling of wake $K_{mn} \times K_n$ vortex ring panels are used, with K_{mn} is stream-wise and K_n in span-wise directions.

The aerodynamics equations based on the vortex lattice model as given in [9] are in discrete time domain as follows:

$$A\Gamma^{t+1} + B\Gamma^t = w_d^{t+1} \quad (14)$$

where A and B are aerodynamics matrices and w_d is the downwash arises from the unsteady motion of the wing. Γ is a vector containing vortex strengths. Behbahani-Nejad et al. [11] investigation show the aerodynamics equation in the form of (14) has some zero eigenvalues which makes some problem in constructing the reduced order aerodynamics model. They modified this model and named it as the modified vortex lattice model. In according to their work, Eq.(14) can be partitioned as:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \Gamma_{wing} \\ \Gamma_{wake} \end{Bmatrix}^{t+1} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{Bmatrix} \Gamma_{wing} \\ \Gamma_{wake} \end{Bmatrix}^t = \begin{Bmatrix} w_d \\ 0 \end{Bmatrix}^{t+1} \quad (15)$$

where Γ_{wing} and Γ_{wake} are the wing and wake vortices, respectively. For vortex lattice aerodynamics model A_{21} , B_{11} and B_{12} are equal to zero and A_{22} is a unity matrix. Hence Eq. (15) can be rewritten as:

$$A_{11}\Gamma_{wing}^{t+1} + A_{12}\Gamma_{wake}^{t+1} = w_d^{t+1} \quad (16)$$

$$\Gamma_{wake}^{t+1} + B_{21}\Gamma_{wing}^t + B_{22}\Gamma_{wake}^t = 0 \quad (17)$$

With calculating Γ_{wing}^t from Eq. (16) in terms of Γ_{wake}^t

and w_d^t and substituting into Eq. of (17), the following equation will be obtained:

$$\Gamma_{wake}^{t+1} + [B_{22} - B_{21}A_{11}^{-1}A_{12}]\Gamma_{wake}^t = -B_{21}A_{11}^{-1}w_d^t \quad (18)$$

2.2 Continuous Aerodynamics Model

Since centre manifold theory is well developed in the continuous time domain, but the presented vortex lattice

aerodynamics mode is in discrete time domain, hence it is necessary to convert the related equation to the continuous time form.

From Taylor series:

$$\Gamma_{wake}^{t+1} = \Gamma_{wake}^t + \left(\frac{d\Gamma_{wake}^t}{dt} \right) dt_1 + \dots \quad (19)$$

By substituting Eq.(19) into (18), the following equation will be obtained,

$$\begin{aligned} \frac{d\Gamma_{wake}^t}{dt} dt_1 + [I + B_{22} - B_{21}A_{11}^{-1}A_{12}]\Gamma_{wake}^t \\ = -B_{21}A_{11}^{-1}w_d^t \end{aligned} \quad (20)$$

With dropping the superscript t from Eqs.(20) and (16), the aerodynamics equations in continuous time domain are obtained and given as follows:

$$\Gamma_{wing} + A_{11}^{-1}A_{12}\Gamma_{wake} = A_{11}^{-1}w_d \quad (21)$$

$$\dot{\Gamma}_{wake} + \frac{I + B_{22} - B_{21}A_{11}^{-1}A_{12}}{dt_1}\Gamma_{wake} = -\frac{B_{21}A_{11}^{-1}}{dt_1}w_d \quad (22)$$

The step time dt_1 can be chosen in terms of aerodynamics grid, as $dt_1 = c_r / K_m / V_\infty$, where V_∞ can be assigned to be $1 m/s$.

Aerodynamics calculations need a lot of mesh grids which increase computation cost. Reduced order aerodynamics model can be used for suppressing these high calculations. Hall presented the eigenanalysis method for constructing reduced order model of the unsteady flow [9]. This method is modified by Behbahani-Nejad *et.al.* [11]. As proposed in [11], the reduced order aerodynamics model can be obtained from the equation governed to the wake vortices Eq.(22). Therefore Γ_{wing} can be obtained from Eq.(21) in terms of this reduced aerodynamics model. This reduced aerodynamics model is as follows [6 and 9]:

$$\Gamma_{wake} = X\bar{C}_{wake} \quad (23)$$

$$\frac{d\bar{C}_{wake}}{d\tau} - \lambda\bar{C}_{wake} = -Y^T \frac{B_{21}A_{11}^{-1}}{dt_1}w_d \quad (24)$$

Where X and Y are right and left eigenvectors of the homogenous part of Eq.(22) and λ is a vector containing related eigenvalues. The \bar{C}_{wake} is modal wake vortex strength. The aerodynamics force on each panel [14] and related generalized aerodynamics work are given in the following form [10]:

$$\Delta p_{ij} = \rho \left[U_\infty \frac{\Gamma_{wing}^{i,j} - \Gamma_{wing}^{i-1,j}}{\Delta c_{ij}} + \frac{\partial \Gamma_{wing}^{i,j}}{\partial t} \right] \quad (25)$$

$$Q_{Aero}^{mn} = \frac{c^4}{6hD} \iint_A \Delta p \phi_m \phi_n d\bar{x} d\bar{y} \quad (26)$$

where ϕ_m and ϕ_n are admissible mode functions for describing transverse displacement and c , h and D are chord, thickness and flexural stiffness of wing, respectively.

3. CENTRE MANIFOLD THEORY

The centre manifold reduction include following steps [16]:

- Identification of the bifurcation point (linear flutter velocity U_F).
- Substituting the free-stream of U_∞ with flutter and difference velocity of $U = U_\infty - U_F$, and rearranging the equation of motion in terms of difference velocity of U .
- Separating the dynamics of the system in two parts, a part with linear hyperbolic eigenvalues, and the other part with linear negative real part.
- Calculation of the centre manifold. For this purpose successive approximation of the centre manifold can be obtained from solving a system of partial differential equations. In this step, the difference velocity of U is not important.
- Reducing dynamics of the system to the centre manifold. For this purpose the obtained centre manifold is used in the dynamics of the .The difference velocity of U is included in this section.

According to the centre manifold theory, at first the nonlinear system must be transformed to Jordan form as follows:

$$\dot{w} = J_0 w + f_1(w, v) \quad (27)$$

$$\dot{v} = J_1 v + f_2(w, v) \quad (28)$$

In which J_0 and J_1 contain purely imaginary and real negative eigenvalues. The centre manifold of $v = V(w)$ is defined as follow:

$$v = V(w), \quad V(0) = 0, \quad \frac{\partial V_j(0)}{\partial w_i} = 0, \quad (29)$$

$$(i = 1, 2, \dots, n_0; \quad j = n_0 + 1, \dots, n)$$

where n_0 and n are dimension of J_0 and total system. The centre manifold will satisfy the following condition:

$$D_w V(w) [J_0 w + f_1(w, V(w))] = J_1 V(w) + f_2(w, V(w)) \quad (30)$$

The analytical solution of the Eq.(30) is not possible, but a polynomial approximation of this equation can be obtained.

4. PRESENTATION OF RESULTS

An aluminum cantilevered plate with aspect ratio of $AR=1$, $L=c=0.3$ m, thickness of $h=0.001$ m, and Poisson's ratio of $\nu = 0.3$ is used as a structural model. For aerodynamics modeling, 800 vortex ring panels, with $K_m=20$, $K_n=20$, and $K_{mm}=140$ elements are used. Twenty mode functions are used for describing in-plane displacements, ten in stream-wise and two in span-wise. The transverse displacement is modeled with eight mode functions, four in span-wise and two in stream-wise [5].

4.1 Aerodynamics and Aeroelastic Results

At first, the validity of aerodynamics model, in continuous time form is investigated. For this purpose the non-dimensional lift coefficient of wing in plunging motion calculated from discrete and continuous form of the full order modified vortex lattice aerodynamic models (Eqs. 18 and 22) are shown in Fig. 2. In this figure, results obtained from discrete and continuous time equations are shown with solid and dash lines, respectively. As seen from this figure, there is good conformity between these results.

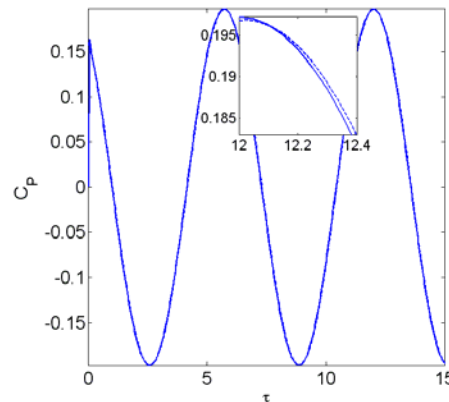


Figure 2 Lift coefficient in plunging motion of wing with full order discrete (solid line) and continuous time (dash line) modified vortex lattice models.

Next, the performance of the continuous time reduced order modified vortex lattice models is investigated. For this purpose, non-dimensional lift coefficient calculated for a plunging motion of wing for full and reduced order models (Eqs. 22 and 24) are shown in Figs. 3. From these figures, with increasing the number of dominate modes, exact prediction of the lift coefficients are obtained (Number of dominate modes for each reduced order models are given by word of DM in figures). Hence the proposed reduced order based on continuous modified vortex lattice model has good conformity in comparison with discrete counterpart.

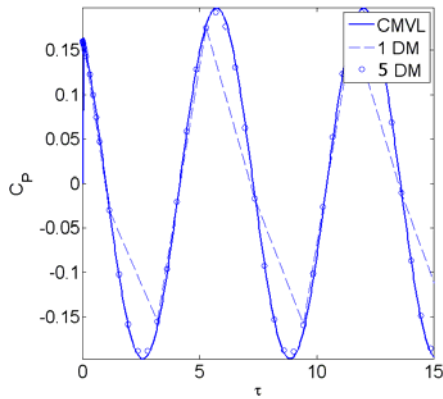


Figure 3 Lift coefficient in plunging motion of wing with full and different reduced orders continuous time modified vortex lattice models.

After examinations of the continuous time modified vortex lattice model for simple calculations, the aeroelastic characteristics of the model are considered here.

With excluding the nonlinear structural term of F_N^{ij} from Eq.(13), and its combination with Eqs. 22 and 24 linear aeroelastic models will be obtained. Calculation of eigenvalues of these linear equations with increasing free stream velocity, determines the stability of the linear aeroelastic system. Figs. 4 and 5 show these eigenvalues for continuous time full and reduced order modified aerodynamics models, respectively. There are two intersections of $\text{Re}(\lambda_i)$ with the velocity axis. One at $U_F = 42\text{m/s}$ for critical flutter velocity and the other is $U_d = 54.3\text{m/s}$ for divergent velocity. As seen from these figures, results obtained from reduced order aerodynamics model are similar with full state aerodynamics model. The results of this section are in complete agreement with results given by [10].

Now the validity of the nonlinear aeroelastic equation of motion in full and reduced order aerodynamic cases are investigated (Eqs. 13, 22 and 24). For this purpose, the transverse displacement of the aeroelastic model with full and reduced order aerodynamics models at a velocity of 60m/s (greater than the flutter velocity) is shown in Fig.6. As seen from Fig.6, the nonlinear aeroelastic model at this velocity undergoes a limit cycle oscillation and the results obtained from full and reduced order aerodynamics models are completely the same. This result is in conformity to the result given in [10].

4.2 Centre Manifold Results

After investigating the validity of the aeroelastic model, the results obtained by applying the centre manifold method are presented.

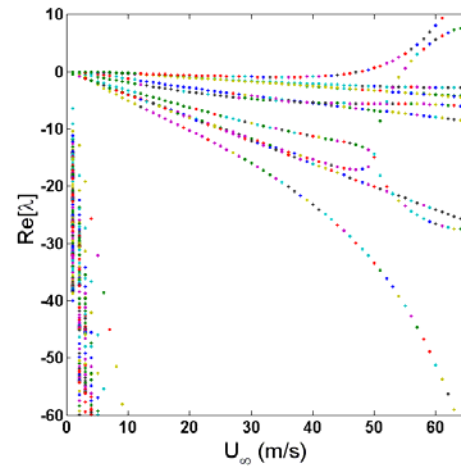


Figure 4 Eigenvalue solution of open loop linear aeroelastic model with full order continuous modified vortex lattice model.

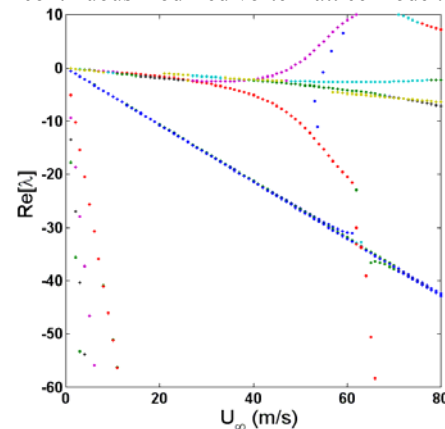


Figure 5 Eigenvalue solution of open loop linear aeroelastic model with reduced order modified continuous vortex lattice model.

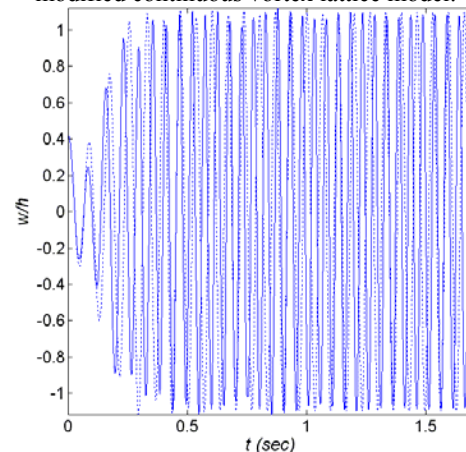


Figure 6 Transverse displacement at 60m/s with full (-) and reduced (- -) order continuous time modified aerodynamics model.

4.2.1 Zero Order Centre Manifold

At first, the results obtained from zero order approximation of the centre manifold are presented. With

this approximation all dynamics which are corresponded to J_1 dynamics of Eqs.(27 and 28) will be ignored and only the J_0 dynamics with related variables are retained in the approximation. The obtained zero order centre manifold is given in appendix 6-a.

A comparisons between aeroelastic responses of the original aeroelastic system (obtained by fourth order Runge-Kutta method) [10] and that obtained from zero order centre manifold approximation is shown in Figs.(7). As seen from this figure, there is relatively good conformity between these two results and this simple approximation present very good approximation of this highly nonlinear system.

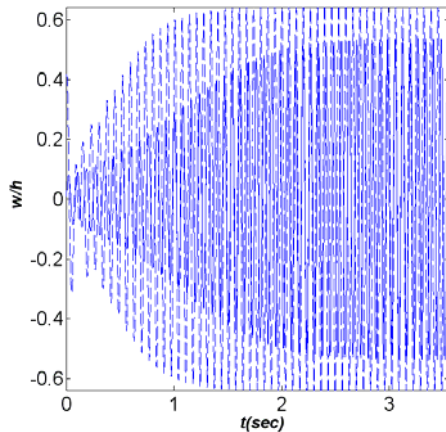


Figure 7 Limit cycle prediction by zero order manifold at 50m/s.

4.2.2 Third Order Centre Manifold

After using a zero order centre manifold approximation it is reasonable to incorporate second order manifold, but because of the form of the nonlinearity presented in this aeroelastic model these manifolds are not applicable here. Hence here a third order approximation of the centre manifold is used in continuo. In this case, the reduced order system will be in the form given in appendix 6-b. The limit cycle predictions obtained from these manifolds and their comparison with those obtained from time response analysis of Ref.[10] are shown in Figs.(8) and (9).

In [6] Grizedzinski investigated limitations in the application of the polynomial approximation of the centre manifold for predicting limit cycle amplitudes. He showed formal power series expansion used in the method of centre manifold reduction will diverge or cannot predict the occurrence of a stable limit cycle for some cases.

In continuo higher order centre manifold will be used for investigating this limitation. For this purpose a fifth order manifold is used.

4.2.3 Fifth Order Centre Manifold

For fifth order manifolds, the reduced system will be in the form given in appendix 6-b.

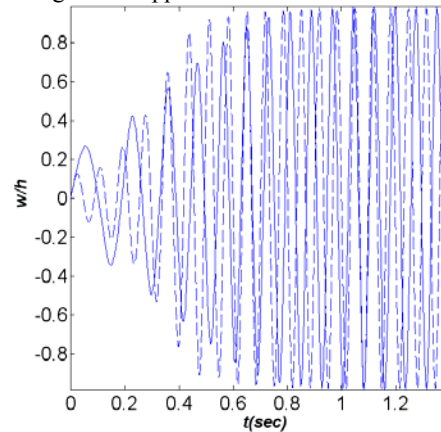


Figure 8 Limit cycle prediction by third order manifold (dash line) and original aeroelastic system (solid line) at 55m/s.

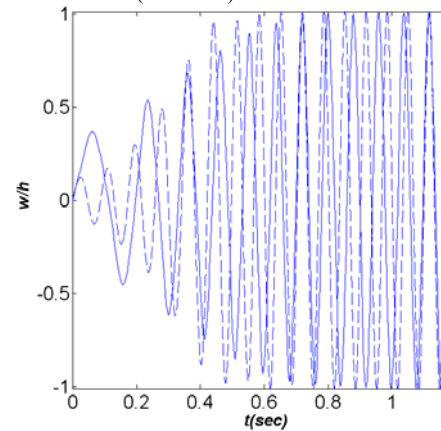


Figure 9 Limit cycle prediction by third order manifold (dash line) and original aeroelastic system (solid line) at 60m/s.

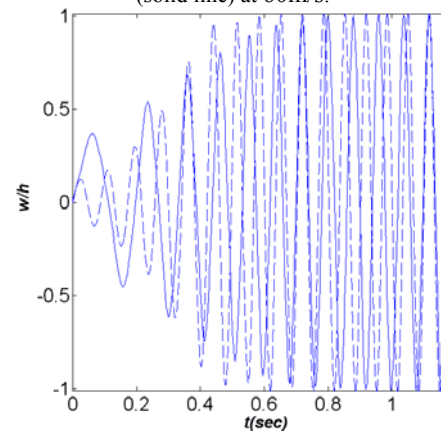


Figure 10 Limit cycle prediction by fifth order manifold (dash line) and original aeroelastic system (solid line) at 60m/s.

The aeroelastic response obtained from fifth order manifold approximation and its comparison with those obtained from original aeroelastic equations are shown in Figs.(10). As seen from this figure, there are very good approximation between two results. Also, these results are in agreement with those results obtained from third order manifold. These results show in current aeroelastic model the limitation stated by Grizedzinski [6] is not presented.

The bifurcation diagram obtained from zero, third and fifth order centre manifold and its comparison with the bifurcation diagram of the original aeroelastic [10] system is shown in Fig.11. As seen from this figure, zero order centre manifold has relatively good accuracy in comparison with higher order manifolds of third and fifth orders, and with increasing manifold orders convergence of the results are obtained.

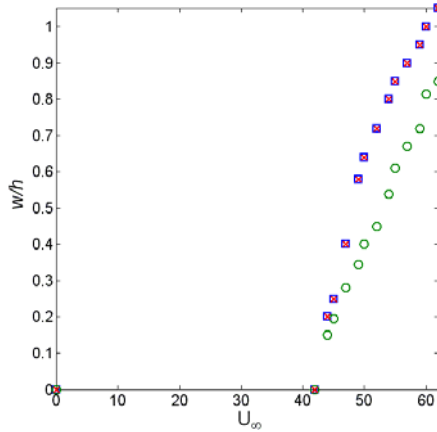


Figure 11 Comparison between different bifurcation diagrams of zero (o), third (x) and fifth (□) order centre manifolds.

5. CONCLUDING REMARKS

In this study, the centre manifold is used for predicting limit cycle amplitude of a highly nonlinear system. Zero, third and fifth order manifolds are used for this purpose and since second and fourth order manifold does not give proper results, they are excluded from calculations. Results obtained from calculations show, the zero order manifold is a nearly good approximation of this highly nonlinear system, and results obtained from higher order manifold, i.e., third and fifth order manifolds give reduced models which completely describe the original very highly nonlinear model. Results obtained from this study can be used for further analysis such as determination of the domain of the attraction of limit cycles and controller design.

6. APPENDIX

a) Reduced model by zero order centre manifold

$$\begin{aligned} \dot{\bar{w}}_1 &= -5.5803\bar{w}_2 + U(0.010645\bar{w}_1 + 0.29417\bar{w}_2) + U^2(0.10535 \\ &\quad \times 10^{-3}\bar{w}_1 + 0.30252 \times 10^{-2}\bar{w}_2) - 4.1897 \times 10^{-5}\bar{w}_1^3 - 0.0019018 \\ &\quad \bar{w}_1^2\bar{w}_2 - 0.030496\bar{w}_1\bar{w}_2^2 - 0.21827\bar{w}_2^3 \\ \dot{\bar{w}}_2 &= 5.5803\bar{w}_1 + U(0.0040673\bar{w}_1 + 0.054743\bar{w}_2) + U^2(2.5897 \\ &\quad \times 10^{-5}\bar{w}_1 + 0.53734 \times 10^{-3}\bar{w}_2) - 6.6975 \times 10^{-6}\bar{w}_1^3 \\ &\quad - 0.27609 \times 10^{-3}\bar{w}_1^2\bar{w}_2 - 0.45355 \times 10^{-2}\bar{w}_1\bar{w}_2^2 \\ &\quad - 0.032114\bar{w}_2^3 \end{aligned}$$

b) Reduced model by third order centre manifold

$$\begin{aligned} \dot{\bar{w}}_1 &= 10^{-2}(0.1231U + 8.5548 \times 10^{-4}U^2)\bar{w}_1^3 - 10^{-3}(1.5978 \times 10^{-3}U^2\bar{w}_2 \\ &\quad + 0.23612U\bar{w}_2)\bar{w}_1^2 + (1.1875 \times 10^{-5}U^2\bar{w}_2^2 + 0.1742 \times 10^{-2}U\bar{w}_2^2 \\ &\quad + 0.010645U + 0.105354 \times 10^{-2}U^2)\bar{w}_1 - 0.0030252U^2\bar{w}_2 + 5.5803\bar{w}_2 \\ &\quad - 0.00017544U\bar{w}_2^3 - 0.29417U\bar{w}_2 - 9.9827 \times 10^{-7}U^2\bar{w}_2^3 - 1.7769 \\ &\quad \times 10^{-6}\bar{w}_1^9 + 4.2462 \times 10^{-7}\bar{w}_2\bar{w}_2^8 + (-3.9592 \times 10^{-8} - 6.986 \times 10^{-6} \\ &\quad \bar{w}_2^2)\bar{w}_1^7 + 10^{-6}(3.1238 \times 10^{-1}\bar{w}_2 + 1.7769\bar{w}_2^3)\bar{w}_1^6 + 10^{-6}(-2.521 \\ &\quad \times 10^{-4} - 9.2184\bar{w}_2^4 - 1.5278 \times 10^{-3}\bar{w}_2^2)\bar{w}_1^5 + 10^{-6}(2.4687 \times 10^{-2} \\ &\quad \bar{w}_2^5 + 3.2022 \times 10^{-3}\bar{w}_2 + 0.8188\bar{w}_2^3)\bar{w}_1^4 + 10^{-6}(-1.1713 \times 10^{-2}\bar{w}_1^2 \\ &\quad - 4.1705\bar{w}_1^6 - 1.6205 \times 10^{-6} - 2.0684 \times 10^{-3}\bar{w}_2\bar{w}_1^3 + 10^{-6}(4.9873 \\ &\quad \times 10^{-3}\bar{w}_2^3 + 1.1423\bar{w}_2^7 + 5.3994 \times 10^{-1}\bar{w}_2^5 + 2.9726 \times 10^{-5}\bar{w}_2)\bar{w}_1^2 \\ &\quad + 10^{-7}(-2.0413 \times 10^{-3}\bar{w}_2^2 - 1.4818 \times 10^{-1}\bar{w}_2^4 - 1.1597 \times 10^{-8}\bar{w}_2^8 \\ &\quad - 9.7607 \times 10^{-1}\bar{w}_2^6)\bar{w}_1 + 1.5172 \times 10^{-9}\bar{w}_2^5 + 4.1713 \times 10^{-9}\bar{w}_2^9 \\ &\quad + 4.895 \times 10^{-9}\bar{w}_2^7 + 4.7899 \times 10^{-10}\bar{w}_2^3 \end{aligned}$$

$$\begin{aligned} \dot{\bar{w}}_2 &= -(0.42441 \times 10^{-3}U + 1.6534 \times 10^{-6}U^2)\bar{w}_1^3 + (3.1218 \times 10^{-7} \\ &\quad U^2\bar{w}_2 + 7.3689 \times 10^{-5}U\bar{w}_2)\bar{w}_1^2 + (-5.5803 - 2.3347 \times 10^{-6}U^2 \\ &\quad \bar{w}_2^2 - 6.4646 \times 10^{-4}U\bar{w}_2^2 - 0.40673 \times 10^{-2}U - 2.5897 \times 10^{-5}U^2 \\ &\quad)\bar{w}_1 + 0.53734 \times 10^{-3}U^2\bar{w}_2 + 6.8787 \times 10^{-5}U\bar{w}_2^3 + 0.054743 \\ &\quad U\bar{w}_2 + 4.1123 \times 10^{-8}U^2\bar{w}_2^3 - 1.7769 \times 10^{-6}\bar{w}_1^9 + 4.2462 \times 10^{-7} \\ &\quad \bar{w}_2\bar{w}_1^8 - (6.986 \times 10^{-6} + 6.986 \times 10^{-6}\bar{w}_2^2)\bar{w}_1^7 + (3.1238 \times 10^{-8} \\ &\quad \bar{w}_2 + 1.7769 \times 10^{-6}\bar{w}_2^3)\bar{w}_1^6 - (9.2184 \times 10^{-6}\bar{w}_2^4 + 2.521 \times 10^{-10} \\ &\quad + 1.5278 \times 10^{-7}\bar{w}_2^2)\bar{w}_1^5 + (2.4687 \times 10^{-6}\bar{w}_1^5 + 3.2022 \times 10^{-9} \\ &\quad \bar{w}_2 + 8.188 \times 10^{-7}\bar{w}_2^3)\bar{w}_1^4 + (-1.1713 \times 10^{-8}\bar{w}_2^2 - 1.6205 \times 10^{-12} \\ &\quad - 4.1705 \times 10^{-6}\bar{w}_2^6 - 2.0684 \times 10^{-7}\bar{w}_2^4)\bar{w}_1^3 + (4.9873 \times 10^{-9} \\ &\quad \bar{w}_2^3 + 1.1423 \times 10^{-6}\bar{w}_2^7 + 5.3994 \times 10^{-7}\bar{w}_2^5 + 2.9726 \times 10^{-11} \\ &\quad \bar{w}_2)\bar{w}_1^2 - (2.0413 \times 10^{-10}\bar{w}_2^2 + 1.4818 \times 10^{-8}\bar{w}_2^4 + 1.1597 \\ &\quad \times 10^{-7}\bar{w}_2^8 + 9.7607 \times 10^{-8}\bar{w}_2^6)\bar{w}_1 + 1.5172 \times 10^{-9}\bar{w}_2^5 \\ &\quad + 4.1713 \times 10^{-9}\bar{w}_2^9 + 4.895 \times 10^{-9}\bar{w}_2^7 + 4.7899 \times 10^{-10}\bar{w}_2^3 \end{aligned}$$

c) Reduced model by fifth order centre manifold

$$\begin{aligned} \bar{w}_1 = & (1.858 \times 10^{-6} U^2 + 2.2264 \times 10^{-4} U) \bar{w}_1^5 + (-3.5147 \times 10^{-5} U \bar{w}_2 \\ & - 3.0147 \times 10^{-7} U^2 \bar{w}_2) \times \bar{w}_1^4 + (2.5 \times 10^{-5} U^2 + 4.8854 \times 10^{-4} U \bar{w}_2^2 \\ & + 3.9461 \times 10^{-6} U^2 \bar{w}_2^2 + 3.0261 \times 10^{-3} U) \bar{w}_1^3 + (-5.0002 \times 10^{-4} U \bar{w}_2 \\ & + 2.9892 \times 10^{-7} U^2 \bar{w}_2^3 - 4.2322 \times 10^{-7} U^2 \bar{w}_2 + 3.9924 \times 10^{-5} U \bar{w}_2^3) \\ & \bar{w}_1^2 + (9.5499 \times 10^{-3} + 3.4888 \times 10^{-5} U^2 \bar{w}_2^2 + 2.0337 \times 10^{-6} U^2 \bar{w}_2^4 \\ & + 1.0535 \times 10^{-4} U^2 + 2.4296 \times 10^{-4} U \bar{w}_2^4 + 1.0645 \times 10^{-2} U + 4.1494 \\ & \times 10^{-3} U \bar{w}_2^2) \bar{w}_1 + 4.7817 \times 10^{-7} U^2 \bar{w}_2^5 + 5.5803 \bar{w}_2 + 5.6654 \times 10^{-5} \\ & U \bar{w}_2^5 - 3.0252 \times 10^{-3} U^2 \bar{w}_2 - 5.8175 \times 10^{-6} U^2 \bar{w}_2^3 - 6.7364 \times 10^{-4} \\ & U \bar{w}_2^3 - 0.29417 U \bar{w}_2 - 7.4246 \times 10^{-14} \bar{w}_1^{15} + 1.0266 \times 10^{-12} \bar{w}_1^{14} \bar{w}_2 \\ & + (-2.9526 \times 10^{-12} \bar{w}_2^2 + 5.2833 \times 10^{-13}) \bar{w}_1^{13} + (-2.2362 \times 10^{-12} \bar{w}_2 \\ & - 2.694 \times 10^{-12} \bar{w}_2^3) \bar{w}_1^{12} + (-9.3579 \times 10^{-9} - 4.5845 \times 10^{-12} \bar{w}_2 \\ & - 1.1522 \times 10^{-11} \bar{w}_2^2) \bar{w}_1^{11} + (2.5911 \times 10^{-11} \bar{w}_2^3 - 6.0056 \times 10^{-9} \bar{w}_2 \\ & - 9.637 \times 10^{-12} \bar{w}_2^5) \bar{w}_1^{10} + (-4.2435 \times 10^{-8} \bar{w}_2^2 + 3.48 \times 10^{-12} \bar{w}_2^6 \\ & - 2.9706 \times 10^{-7} - 9.7819 \times 10^{-13} \bar{w}_2^4) \bar{w}_1^9 + (-2.9073 \times 10^{-8} \bar{w}_2^3 \\ & + 9.3266 \times 10^{-11} \bar{w}_2^5 - 1.3198 \times 10^{-7} \bar{w}_2 - 3.6399 \times 10^{-13} \bar{w}_2^7) \bar{w}_1^8 \\ & + (-1.0581 \times 10^{-6} \bar{w}_2^2 - 3.4062 \times 10^{-6} - 7.0411 \times 10^{-8} \bar{w}_2^4 \\ & + 3.813 \times 10^{-12} \bar{w}_2^8 - 2.5236 \times 10^{-11} \bar{w}_2^6) \bar{w}_1^7 + (-5.3782 \times 10^{-7} \\ & \bar{w}_2^3 + 3.3068 \times 10^{-5} \bar{w}_2 - 5.3319 \times 10^{-8} \bar{w}_2^5 - 1.9538 \times 10^{-11} \bar{w}_2^7 \\ & + 6.9856 \times 10^{-12} \bar{w}_2^9) \bar{w}_1^6 + (-3.8649 \times 10^{-4} \bar{w}_2^2 - 5.0746 \times 10^{-8} \\ & \bar{w}_2^6 - 1.6931 \times 10^{-5} + 4.1067 \times 10^{-11} \bar{w}_2^8 - 2.3112 \times 10^{-12} \bar{w}_2^{10} \\ & - 1.1846 \times 10^{-6} \bar{w}_2^4) \bar{w}_1^5 + (-4.1143 \times 10^{-8} \bar{w}_2^7 + 3.3382 \times 10^{-12} \\ & \bar{w}_2^{11} + 8.5814 \times 10^{-5} \bar{w}_2^3 + 5.3585 \times 10^{-4} \bar{w}_2 - 6.8796 \times 10^{-7} \\ & \bar{w}_2^5 - 7.776 \times 10^{-11} \bar{w}_2^9) \bar{w}_1^4 + (-2.6767 \times 10^{-12} \bar{w}_2^{12} - 8.9017 \\ & \times 10^{-4} \bar{w}_2^4 + 2.4276 \times 10^{-11} \bar{w}_2^{10} - 4.1897 \times 10^{-5} - 1.7219 \times \\ & 10^{-8} \bar{w}_2^8 - 4.3249 \times 10^{-7} \bar{w}_2^6 - 5.4668 \times 10^{-3} \bar{w}_2^2) \bar{w}_1^3 + \\ & (3.6219 \times 10^{-6} \bar{w}_2^5 - 1.751 \times 10^{-11} \bar{w}_2^{11} - 1.0475 \times 10^{-8} \bar{w}_2^9 \\ & + 1.6174 \times 10^{-3} \bar{w}_2^3 + 7.1104 \times 10^{-14} \bar{w}_2^{13} + 1.9018 \times 10^{-3} \bar{w}_2 \\ & - 2.7088 \times 10^{-7} \bar{w}_2^7) \bar{w}_1^2 + (-2.5501 \times 10^{-9} \bar{w}_2^{10} - 3.9881 \times \\ & 10^{-4} \bar{w}_2^6 + 1.4119 \times 10^{-11} \bar{w}_2^{12} - 5.9189 \times 10^{-13} \bar{w}_2^{14} - 3.0496 \\ & \times 10^{-2} \bar{w}_2^2 - 7.0582 \times 10^{-3} \bar{w}_2^4 - 3.0949 \times 10^{-8} \bar{w}_2^8) \bar{w}_1 \\ & - 9.0552 \times 10^{-11} \bar{w}_2^{11} - 3.0514 \times 10^{-14} \bar{w}_2^{15} - 9.7985 \times 10^{-5} \\ & \bar{w}_2^7 + 1.4807 \times 10^{-9} \bar{w}_2^9 + 0.21827 \bar{w}_2^3 + 2.906 \times 10^{-12} \bar{w}_2^{13} \\ & + 6.992 \times 10^{-4} \bar{w}_2^5 \end{aligned}$$

$$\begin{aligned} \bar{w}_2 = & (-3.2175 \times 10^{-7} U^2 - 3.8467 \times 10^{-5} U) \bar{w}_1^5 + (9.4923 \times 10^{-7} \\ & U \bar{w}_2 + 1.367 \times 10^{-8} U^2 \bar{w}_2) \times \bar{w}_1^4 + (-4.4271 \times 10^{-6} U^2 - 8.9419 \\ & \times 10^{-5} U \bar{w}_2^2 - 7.5612 \times 10^{-7} U^2 \bar{w}_2^2 - 5.2048 \times 10^{-4} U) \times \bar{w}_1^3 \\ & + (1.2557 \times 10^{-4} U \bar{w}_2 + 1.7406 \times 10^{-8} U^2 \bar{w}_2^3 + 1.1587 \times 10^{-6} \\ & U^2 \bar{w}_2 - 4.5079 \times 10^{-7} U \bar{w}_2^3) \bar{w}_1^2 + (-5.5803 - 5.671 \times 10^{-6} U^2 \bar{w}_2^2 \\ & - 3.4238 \times 10^{-7} U^2 \bar{w}_2^4 - 2.5897 \times 10^{-5} U^2 - 4.0312 \times 10^{-5} U \bar{w}_2^4 \\ & - 4.0673 \times 10^{-3} U - 6.636 \times 10^{-4} U \bar{w}_2^2) \bar{w}_1 - 5.5323 \times 10^{-8} U^2 \bar{w}_2^5 \\ & + 9.5499 \times 10^{-3} \bar{w}_2 - 7.3247 \times 10^{-6} U \bar{w}_2^5 + 5.3734 \times 10^{-4} U^2 \bar{w}_2 \\ & + 6.9296 \times 10^{-7} U^2 \bar{w}_2^3 + 7.5882 \times 10^{-5} U \bar{w}_2^3 + 0.054743 U \bar{w}_2 \\ & + 1.034 \times 10^{-14} \bar{w}_1^{15} - 1.2923 \times 10^{-13} \bar{w}_1^{14} \bar{w}_2 + (2.4963 \times 10^{-13} \\ & \bar{w}_2^2 - 1.1171 \times 10^{-14} \bar{w}_1^{13} + (-5.213 \times 10^{-13} \bar{w}_2 + 7.4964 \times 10^{-13} \\ & \bar{w}_2^3) \bar{w}_1^{12} + (1.4305 \times 10^{-9} + 1.0385 \times 10^{-12} \bar{w}_2^4 + 3.0777 \times 10^{-12} \\ & \bar{w}_2^2) \bar{w}_1^{11} + (3.0085 \times 10^{-12} \bar{w}_2^3 + 1.1281 \times 10^{-9} \bar{w}_2 + 1.8295 \times 10^{-12} \\ & \bar{w}_2^5) \bar{w}_1^{10} + (6.5557 \times 10^{-9} \bar{w}_2^2 + 1.1752 \times 10^{-12} \bar{w}_2^6 + 4.6826 \times \\ & 10^{-8} + 5.1238 \times 10^{-12} \bar{w}_2^4) \bar{w}_1^9 + (5.2432 \times 10^{-9} \bar{w}_2^3 - 2.2072 \times \\ & 10^{-12} \bar{w}_2^5 + 2.5914 \times 10^{-8} \bar{w}_2 - 3.237 \times 10^{-13} \bar{w}_2^7) \bar{w}_1^8 + (1.6742 \\ & \times 10^{-7} \bar{w}_2^2 + 5.2606 \times 10^{-7} + 1.08 \times 10^{-8} \bar{w}_2^4 - 8.6152 \times 10^{-13} \bar{w}_2^8 \\ & - 1.5224 \times 10^{-11} \bar{w}_2^6) \bar{w}_1^7 + (1.0169 \times 10^{-7} \bar{w}_2^3 - 4.7812 \times 10^{-5} \\ & \bar{w}_2^4 + 9.4746 \times 10^{-9} \bar{w}_2^5 - 5.8063 \times 10^{-12} \bar{w}_2^7 - 1.6592 \times 10^{-12} \\ & \bar{w}_2^9) \bar{w}_1^6 + (5.4833 \times 10^{-5} \bar{w}_2^2 + 8.1463 \times 10^{-9} \bar{w}_2^6 + 2.5648 \times 10^{-6} \\ & - 5.0086 \times 10^{-12} \bar{w}_2^8 - 1.0603 \times 10^{-12} \bar{w}_2^{10} + 1.9012 \times 10^{-7} \bar{w}_2^4) \bar{w}_1^5 \\ & + (7.2938 \times 10^{-9} \bar{w}_2^7 - 1.7273 \times 10^{-13} \bar{w}_2^{11} - 2.3271 \times 10^{-7} \bar{w}_2^3 \\ & - 8.2792 \times 10^{-5} \bar{w}_2 + 1.255 \times 10^{-7} \bar{w}_2^5 + 3.8005 \times 10^{-13} \bar{w}_2^9) \bar{w}_1^4 \\ & + (7.1964 \times 10^{-14} \bar{w}_2^{12} + 1.1813 \times 10^{-4} \bar{w}_2^4 + 9.9423 \times 10^{-12} \bar{w}_2^{10} \\ & + 6.6975 \times 10^{-6} + 2.9697 \times 10^{-9} \bar{w}_2^8 + 7.1389 \times 10^{-8} \bar{w}_2^6 + 8.4314 \\ & \times 10^{-4} \bar{w}_2^2) \bar{w}_1^3 + (3.7362 \times 10^{-5} \bar{w}_2^5 - 4.1823 \times 10^{-13} \bar{w}_2^{11} + 1.8307 \\ & \times 10^{-9} \bar{w}_2^9 + 1.847 \times 10^{-6} \bar{w}_2^3 + 3.9553 \times 10^{-13} \bar{w}_2^{13} - 2.7609 \times 10^{-4} \\ & \bar{w}_2 + 4.8301 \times 10^{-8} \bar{w}_2^7) \bar{w}_1^2 + (3.7934 \times 10^{-10} \bar{w}_2^{10} + 5.3136 \times 10^{-5} \\ & \bar{w}_2^6 + 1.2633 \times 10^{-12} \bar{w}_2^{12} + 1.5976 \times 10^{-13} \bar{w}_2^{14} + 4.5355 \times 10^{-3} \\ & \bar{w}_2^2 + 1.1325 \times 10^{-3} \bar{w}_2^4 + 5.4709 \times 10^{-9} \bar{w}_2^8) \bar{w}_1 + 1.2257 \times 10^{-11} \\ & \bar{w}_2^{11} + 7.522 \times 10^{-14} \bar{w}_2^{15} + 3.0516 \times 10^{-5} \bar{w}_2^7 + 4.5591 \times 10^{-10} \bar{w}_2^9 \\ & - 0.032114 \bar{w}_2^3 - 2.2633 \times 10^{-12} \bar{w}_2^{13} + 1.1156 \times 10^{-4} \bar{w}_2^5 \end{aligned}$$

7. REFERENCES

Periodicals:

- [1] E. H. Dowell, J. Edwards, and T. W. Strganac, *Nonlinear Aeroelasticity*, *J. Aircraft*, 40 (5) (2003) 857-874.
- [2] Liu, L., Wong, Y. S., and Lee, B.H.K, *Journal of Sound and Vibration* 2002, 234(4), 641-659, Application of the centre manifold theory in non-linear aeroelasticity.
- [3] Shahrzad, P. and Mahzoon, M., *Journal of Sound and Vibration* 2002 256(2), 213-225, Limit cycle flutter of airfoils in steady and unsteady flows.
- [4] Dess, D., Mastroddi, F., and Morino, L., *Journal of Sound and Vibration* 2002 256 (2) 347-365, Limit cycle stability reversal near a Hopf bifurcation with aeroelastic applications.
- [5] Dess, D., Mastroddi, F., *Journal of Fluids and Structures* 2004 19, 765-783, Limit cycle stability reversal via singular perturbation and wing flap flutter.
- [6] Grzedzinski, J., *Journal of Fluids and Structures* 2005, 21, 187-209, Limitation of application of the centre manifold reduction in aeroelasticity
- [7] Qian , D., and Li, W. D., *Aerospace Science and Technology* 2006, 10, 427-434, The flutter of an airfoil with cubic structural and aerodynamic nonlinearities.
- [8] Bi, Q.S., Yu, P., *Journal of Computer and Applied Mathematics* 1999, 102, 195-220, Symbolic computation of normal forms for semi-simple cases.
- [9] K.C. Hall, *AIAA Journal*, 1994, 32 (12) 2426-2423 Eigenanalysis of Unsteady Flows About Airfoils, Cascades, and Wings.
- [10] D. Tang, E.H. Dowell and K.C. Hall, *AIAA Journal*, 1999, 37 (3) 364-371 Limit Cycle Oscillation of a Cantilevered Wing in Low Subsonic Flow.
- [11] M. Behbahani-Nejad, H. Haddadpour, and V. Esfahanian, *Journal of Aircraft* 2005, 42 (4) 882-886, Reduced-Order modeling of unsteady flows without static correction requirement

Books:

- [12] Dowell, E.H., *Aeroelasticity of Plates and Shells*, Kluwer/Noordhoff International Publishing, Leyden, 1975.
- [13] Meirovitch, L., *Principles and Techniques of Vibrations*, Prentice-Hall International INC. 1997.
- [14] J. Katz and A. Plotkin 1991, *Low Speed Aerodynamics, From Wing Theory to Panel Methods*, MCGRAW – HILL ,New York.
- [15] J. Carr 1981 *Application of Centre Manifold Theory*, New York, Springer-Verlag.