Extension of Higher Order Derivatives of Lyapunov Functions in Stability Analysis of Nonlinear Systems

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ABSTRACT

The Lyapunov stability method is the most popular and applicable stability analysis tool of nonlinear dynamic systems. However, there are some bottlenecks in the Lyapunov method, such as need for negative definiteness of the Lyapunov function derivative in the direction of the system's solutions. In this paper, we develop a new theorem to dispense the need for negative definite-ness of Lyapunov function derivative. We introduce new sufficient conditions for asymptotic stability of equilibrium states of nonlinear systems considering some inequalities for the higher order time derivatives of Lyapunov function. If the above-mentioned inequalities are found, then the stability analysis of an equilibrium state is reduced to check the characteristic equation for a controllable canonical form LTI co-system. The poles of co-system are required to be negative real ones. Some examples are presented to demonstrate the approach.

KEYWORDS

Nonlinear dynamic systems, Lyapunov methods, stability analysis.

1. INTRODUCTION

Let the following n-dimensional dynamic system has a ZES, i.e., $\underline{f}(t,0)=0 \quad \forall t \ge 0$ (for the entire abbreviations see the nomenclature).

$$\underline{\dot{x}} = \underline{f}(t,\underline{x}) \quad t \ge 0, \quad \underline{x} \in \mathbb{R}^n \tag{1}$$

The advantage of the Lyapunov method is use of a LF or energy like function. The classical Lyapunov approach for the UAS of ZES of (1) requires existence of a *decrescent* LPDF $v(t,\underline{x})$ whose total time derivative $\dot{v}(t,\underline{x})$ along the solutions of (1) is negative definite. When $\dot{v}(t,\underline{x})$ is negative semi-definite, stability rather than AS follows; but in the case of autonomous systems, the LaSalle's invariance principle may be helpful in proving the AS of ZES. See [1] and [2].

Consider a system with an asymptotically stable ZES with a non-monotone LF candidate $v(t,\underline{x})$. In addition, let us have a *sense* that "*The decrement of the LF candidate is more than its increment*" and $v(t,\underline{x}(t)) \to 0$ as $t \to \infty$. In this case does this LF candidate verify the stability of the system by the Lyapunov theorems? The answer is negative. However, it is somewhat challenging to find the right LF '$v(t,\underline{x})$'.

The work of Narendra et al. in [3] which completed in [4] showed that if there exist a $T>0$ and a function $\gamma \in K$ ($\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, increasing and $\gamma(0)=0$) such that $v(t+T,\underline{x}(t+T)) - v(t,\underline{x}(t)) \le -\gamma(\|\underline{x}(t)\|) < 0, \forall t$ then the ZES is UAS. However, this result is not very interesting, because it needs solutions of system or some bounds for them. The references [5] and [6] introduced a VLF

$$\underline{V}(t,\underline{x}) = [v_1(t,\underline{x}), v_2(t,\underline{x}), \ldots, v_m(t,\underline{x})]^t \tag{2}$$

for the stability analysis. Each $v_i(t,\underline{x})$ in (2) should at least be positive semi-definite and $s(t,\underline{x}) \triangleq \sum_{i=1}^{m} k_i v_i(t,\underline{x})$, $k_i > 0$ is a LF candidate in the Lyapunov direct method.

However, some papers such as [7] and [8] used VLFs in a different way. They need $\underline{\dot{V}}(t,\underline{x}) \le g[\underline{V}(t,\underline{x})]$ component-vise, i.e. $\dot{v}_i(t,\underline{x}) \forall i$ might be an indefinite function. They compared $\underline{V}(t,\underline{x})$ to a new co-system $\underline{\dot{u}}(t) = g[\underline{u}(t)]$, $\underline{u} \in \mathbb{R}^m$. The comparison is valid if the map $g : \mathbb{R}^m \to \mathbb{R}^m$ is of class W (see Appendix), thus $v_i(t,\underline{x}) \le u_i(t) \ \forall i = 1,\ldots,m$ and $\forall t \ge t_0$ for identically same initial conditions, i.e., $v_i(t_0,\underline{x}_0) = u_i(t_0)$. Hence, the ZES of (1) is AS as long as the ZES of the co-system is

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AS.

In [9]-[10] for the first time, use of $\ddot{v}$ along the systems trajectories was reported. Gunderson [11] considered stability analysis of (1) using $v^{(m)}(t,\underline{x}) \le g_m(t,v,\dot{v},\cdots,v^{(m-1)})$ for a LPDF $v(t,\underline{x})$ and for some $m \in \mathbb{N}$, where the higher order time derivatives $v^{(i)}(t,\underline{x})$ are computed along the systems trajectories. This inequality is compared with a new nonlinear co-system $u^{(m)}(t) = g_m(t,u,\dot{u},\cdots,u^{(m-1)})$, called comparison equation. If the ZES of the co-system is AS and the vector field $g[\cdot]$ of the co-system is of class W, then the ZES of (1) is also AS. The co-system cannot be LTI in this method, as indicated in [11] and will be shown later in this paper.

Butz 0 showed the ZES of $\dot{\underline{x}} = \underline{f}(\underline{x})$ is AS if $a_3\dddot{v}(\underline{x}) + a_2\ddot{v}(\underline{x}) + \dot{v}(\underline{x}) < 0$, $\forall \underline{x} \ne 0$ ($a_2 \ge 0$ and $a_3 \ge 0$). A marginally stable LTI co-system $a_3\dddot{u}(t) + a_2\ddot{u}(t) + \dot{u}(t) = 0$ and the relationship $v(\underline{x}(t)) \le u(t)$ were used in their proof. The extension of this proof fails for the use of higher order derivatives, i.e., $\sum_{i=1}^{m} a_i v^{(i)}(\underline{x}) < 0$ for $m > 3$, because the related LTI co-system $\sum_{i=1}^{m} a_i u^{(i)}(t) = 0$ ($a_i \ge 0$) is not necessarily marginally stable and the relationship $v(\underline{x}(t)) \le u(t)$ is not true in the case of $m > 3$. A recent work on stability analysis of discrete time systems using non-monotone LF candidates is reported in [13].

In this paper, asymptotical stability of ZES of (1) is proved using a new condition on higher order derivatives of LF candidates. Our method uses $\sum_{i=0}^{m} a_i v^{(i)}(t,\underline{x}) \le 0$ for some $m \in \mathbb{N}$, while the characteristic equation $\sum_{i=0}^{m} a_i s^i = 0$ should only have negative real roots.

Our method uses a special VLF, e.g., $v_i(t,\underline{x}) \triangleq v^{(i-1)}(t,\underline{x})$, $i = 1,\ldots,m$ in (2). However, only the first component of this VLF is LPDF and the other components may be indefinite. This is the main difference with [5]-[8]. Comparing with [11], the new method uses an LTI co-system, with much simpler stability conditions. Comparing with 0, our approach could be implemented with higher order time derivatives and is applicable to time varying systems. On the other hand, our method is improved to consider the case where the nonlinear system is not smooth enough and the higher order time derivatives $v^{(i)}(t,\underline{x})$ are not well-defined. We use the stability definitions of Lyapunov stability theory, which are used in the Lyapunov theorems. See [1] and [2].

This paper is organized as follows: The problem statement and the key ideas are presented in Section 2. While our approach and some preliminary results are presented in Section 3. The main theorem for stability analysis is given in Section 4. Some examples are settled to apply our method in Section 5. Concluding remarks are drawn in Section 6.

2. The Problem Statement

Suppose $\underline{f}(t,\underline{x})$ in (1) and a LPDF $v(t,\underline{x})$ are smooth enough such that $\dot{v}(t,\underline{x}),\ddot{v}(t,\underline{x}),\ldots,v^{(m)}(t,\underline{x})$ along the solutions of (1) for some $m \in \mathbb{N}$ are well-defined and are computed iteratively, i.e.,

$$v^{(i)}(t,\underline{x}) \triangleq \partial v^{(i-1)}/\partial t + \underline{f}(t,\underline{x})^T[\partial v^{(i-1)}/\partial \underline{x}] \quad (3)$$

and $\dot{v}(t,\underline{x})$ is not negative definite. Gunderson [11] used comparison method for $v^{(m)}(t,\underline{x}) \le g_m(t,v,\dot{v},\cdots,v^{(m-1)})$. The method uses a co-system $u^{(m)}(t) = g_m(t,u,\dot{u},\cdots,u^{(m-1)})$ with an asymptotically stable ZES and a class W vector field. Then $v^{(i)}(t,\underline{x}) \le u^{(i)}(t), \forall i = 0,1,\ldots,(m-1)$, $\forall t \ge t_0$ is implied when both initial conditions are identically equal. Since the ZES of the co-system is AS thus the ZES of (1) is also AS.

We are interested in LTI co-systems because the stability of these systems can be easily verified. However, the method of [11] fails using LTI co-systems, because the class W condition for $u^{(m)}(t) = -a_{m-1}u^{(m-1)}(t) - \cdots - a_1\dot{u}(t) - a_0u(t)$ implies $-a_i \ge 0$ for $i = 0,1,\ldots,m-2$ (See Appendix). Thus the characteristic equation is not Hurwitz and the co-system is not stable, so the Gunderson's method does not work using LTI co-systems.

We believe that the class W condition in the Gunderson's method is too restrictive, so we omit it in order to preserve the asymptotically stable LTI co-systems. Thus the consequence $v^{(i)}(t,\underline{x}) \le u^{(i)}(t)$ for $i = 0,1,\ldots,m-1$ is no longer held. In our new method, instead of the above relationships '$v^{(i)}(t,\underline{x}) \le u^{(i)}(t)$' only $v(t,\underline{x}) \le u(t)$ for $t \ge t_0$ will be used. Let the following inequality holds:

$$v^{(m)}(t,\underline{x}) + a_{m-1}v^{(m-1)}(t,\underline{x}) + \cdots + a_1\dot{v}(t,\underline{x}) + a_0v(t,\underline{x}) \le 0 \quad (4)$$

For a given trajectory $\underline{x}(t,t_0,\underline{x}_0)$, let us compare $v(t,\underline{x}(t,t_0,\underline{x}_0))$ with another signal $u(t)$ which satisfies the following relationships:

$$u^{(m)}(t) + a_{m-1}u^{(m-1)}(t) + \cdots + a_1\dot{u}(t) + a_0u(t) = 0 \quad (5)$$

$$u^{(i)}(t_0) = v^{(i)}(t_0,\underline{x}_0), \quad i = 0,1,2,\ldots,m-1 \quad (6)$$

with the following characteristic equation

$$s^m + a_{m-1}s^{m-1} + \cdots + a_1s + a_0 = 0 \quad (7)$$

If this polynomial is Hurwitz, then the state vector

$\underline{U}(t)=[u(t),\dot{u}(t),\ldots,u^{(m-1)}(t)]^t$ is exponentially stable and $u(t)\to 0$. Thus we may prove AS of the ZES of (1) if

$$0 \le v(t,\underline{x}) \le u(t) \quad \forall t \ge t_0 \tag{8}$$

However, (8) is not always true; a counterexample is followed.

Example 1: The two signals $u(t)=e^{-t}\cos t$ and $v(t)=u(t)-2e^{-4t}+2e^{-t}\cos t-6e^{-t}\sin t$ satisfy the following relationships:

$$\begin{cases}\ddot{u}(t)+2\dot{u}(t)+2u(t)=0 \quad , \quad \ddot{v}(t)+2\dot{v}(t)+2v(t)\le 0 \\ v(0)=u(0) \quad , \quad \dot{v}(0)=\dot{u}(0)\end{cases} \tag{9}$$

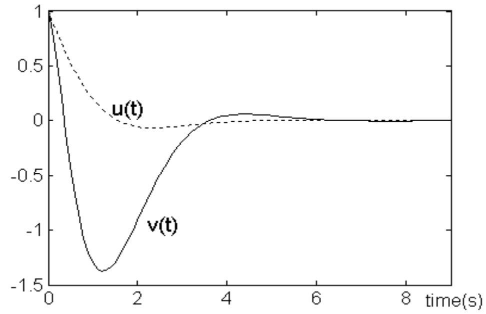

Figure 1: Sketch of $u(t)$ and $v(t)$ in Example 1

but Fig. 1 shows that $v(4)>u(4)$. This happened because the poles of the characteristic equation for $\ddot{v}+2\dot{v}+2v=d$ are complex, and the transient response is under damped. Thus the solution of differential inequality (4) is not generally given as (8), using (5) and (6).■

3. Development Of New Approach

First, the definition of *decrescent-ness* from Lyapunov stability theory is generalized for vector functions. A vector function $\underline{V}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$ is called *decrescent* if there exists $0<r_2\le+\infty$ and $\psi\in K$ such that:

$$\left\|\underline{V}(t,\underline{x})\right\| \le \psi(\|\underline{x}\|) \quad \forall t,\forall \underline{x}\in B_{r_2} \tag{10}$$

It is clear that the case of $r_2=+\infty$ is considered only for global stability. We sometimes refer to this case as *globally decrescent*. Let us consider first the equality part of (4) and develop our approach.

Theorem 1: Let the nonlinear system (1) and a function $v_1(t,\underline{x})$ be smooth enough such that the higher order time derivatives $\dot{v}_1(t,\underline{x}),\ddot{v}_1(t,\underline{x}),\ldots,v_1^{(m)}(t,\underline{x})$ along the solutions of (1) for some $m\in\mathbb{N}$ are well-defined using (3), and

$$v_1^{(m)}(t,\underline{x})+a_{m-1}v_1^{(m-1)}(t,\underline{x})+\cdots+a_0v_1(t,\underline{x})=0\ , \tag{11}$$

while the characteristic polynomial (7) is Hurwitz. Define

$$\underline{V}(t,\underline{x}) \triangleq [v_1(t,\underline{x}),\dot{v}_1(t,\underline{x}),\ldots,v_1^{(m-1)}(t,\underline{x})]^T\ , \tag{12}$$

then we have the followings:

a) If $v_1(t,\underline{x})$ is LPDF, i.e., $v_1(t,\underline{0})=0$ and there exists $r_1>0$ and a function $\phi_1\in K$ such that $\forall\underline{x}\in B_{r_1}$

$$v_1(t,\underline{x}) \ge \phi_1(\|\underline{x}\|) \tag{13}$$

then the ZES of (1) is AS.

b) If $\underline{V}(t,\underline{x})$ is decrescent, while its first component $v_1(t,\underline{x})$ is LPDF, then the ZES of (1) is UAS.

c) If $v_1(t,\underline{x})$ is RU and PDF then the ZES of (1) is GAS.

d) If $\underline{V}(t,\underline{x})$ is globally decrescent, while its first component $v_1(t,\underline{x})$ is RU and PDF, then the ZES of (1) is UGAS.■

Before proving this theorem, let us present a lemma about continuous vector functions and recall another one from Lyapunov stability theory.

Lemma 1: Considering $\underline{V}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$, we have:

i. For every t, $\underline{V}(t,\underline{0})=\underline{0}$ and $\underline{V}(t,\underline{x})$ is continuous at $\underline{x}=\underline{0}$, iff there exists $r_2(t)>0$ and existing function is $\psi_t\in K$ depending on t, such that for any $\underline{x}\in B_{r_2(t)}$:

$$\left\|\underline{V}(t,\underline{x})\right\| \le \psi_t(\|\underline{x}\|) \tag{14}$$

ii. $\underline{V}(t,\underline{0})=\underline{0},\forall t$ and $\underline{V}(t,\underline{x})$ is uniformly continuous at $\underline{x}=\underline{0}$, iff $\underline{V}(t,\underline{x})$ is decrescent.

iii. For every *t*, if $\underline{V}(t,\underline{0})=\underline{0}$ and $\underline{V}(t,\underline{x})$ is continuous everywhere then there exists a function $\psi_t\in K$, such that (14) is satisfied globally. ■

Proof:

The "*if*" parts of (i)-(ii) are clear, thus we prove the "*only if*" parts of (i)-(ii).

The "only if" Part of i); Let for a fixed time t, $\underline{V}(t,\underline{0})=\underline{0}$ and $\underline{V}(t,\underline{x})$ be continuous at $\underline{x}=\underline{0}$. So for each $\varepsilon>0$ there is $\delta(t,\varepsilon)>0$, such that if $\|\underline{x}\|<\delta$ then $\left\|\underline{V}(t,\underline{x})\right\|<\varepsilon$. Define $\gamma_t(r)\triangleq\sup_{\|\underline{x}\|\le r}\left\|\underline{V}(t,\underline{x})\right\|$ over a sufficiently small interval $r\in[0,r_2(t)]$. This function is nondecreasing, continuous at $r=0$ and $\gamma_t(0)=0$, thus existing function is $\psi_t\in K$ such that $\gamma_t(r)\le\psi_t(r)$, $\forall r\ge 0$. In this case (14) is satisfied.

The "only if" Part of ii); Let $\underline{V}(t,\underline{x})$ be zero and uniformly continuous at $\underline{x}=\underline{0}$, so for any $\varepsilon>0$ there is a $\delta(\varepsilon)>0$, such that if $\|\underline{x}\|<\delta$ then $\left\|\underline{V}(t,\underline{x})\right\|<\varepsilon$. Now define $\gamma(r)\triangleq\sup_{\|\underline{x}\|\le r,\forall t}\left\|\underline{V}(t,\underline{x})\right\|$ over a sufficiently small interval $r\in[0,r_2]$. Similar to the previous part existing function is $\psi\in K$ such that $\gamma(r)\le\psi(r)\ \forall r\ge 0$. This function is independent of time *t*, and (10) is satisfied.

Part iii) Let t be a fixed time, since $\underline{V}(t,\underline{x})$ is continuous then the function $\gamma_t(r)=\sup_{\|\underline{x}\|\le r}\|\underline{V}(t,\underline{x})\|$ is defined for all $r\ge 0$ and is continuous. Applying the approach of *Part* i) with $r\in[0,+\infty)$ completes the proof. ■

Lemma 2 [2]: The ZES of (1) is UAS iff there exist a $\beta\in KL$ and $c>0$ independent of t_0, such that

$$\|\underline{x}(t)\|\le\beta(\|\underline{x}(t_0)\|,t-t_0),\quad\forall t\ge t_0,\quad\forall\|\underline{x}(t_0)\|<c \tag{15}$$
.

It is UGAS iff (15) is satisfied for all $x(t_0)$. ■

$\beta(p,t)\in KL$ means that $\beta:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous, strictly increasing with respect to p, strictly decreasing with respect to t, $\beta(0,t)=0$ and $\beta(p,t)\to 0$ as $t\to+\infty$.

Corollary 1: If Lemma 2 is considered with $\beta_{t_0}\in KL$ depending on initial time t_0, then non uniform stabilities of ZES will be concluded, respectively. ■

Proof of Theorem 1:

Consider the LTI co-system (5) with the state vector $\underline{U}(t)\triangleq[u(t),\dot{u}(t),\dots,u^{(m-1)}(t)]^T$. Since the polynomial (7) is Hurwitz, thus the system (5) is globally exponentially stable, and there are $a,b>0$ such that for any initial value $\underline{U}(t_0)$ we have $|u(t)|\le a\|\underline{U}(t_0)\|\exp[-b(t-t_0)]$. The signal $v_1(t,\underline{x}(t))$ along the solutions of (1) satisfies (11), and the above result holds for $v_1(t,\underline{x}(t))$ and $\underline{V}(t,\underline{x}(t))$ defined in (12) along the solutions of (1) and we have:

$$v_1(t,\underline{x})\le a\|\underline{V}(t_0,\underline{x}_0)\|\exp[-b(t-t_0)]\quad\forall t\ge t_0 \tag{16}$$

Note that in this theorem $v_1(t,\underline{0})\equiv 0$, thus $\underline{V}(t,\underline{0})\equiv\underline{0}$.

Part a) $\underline{V}(t_0,\underline{0})=\underline{0}$ and $\underline{V}(t_0,\underline{x}_0)$ is C^1, thus it is continuous at $\underline{x}_0=\underline{0}$. Applying Lemma 1(i) for $\underline{V}(t_0,\underline{x}_0)$ at a fixed t_0, implies (14) holds for t_0. $v_1(t,\underline{x})$ is LPDF, thus by combining the relationships (13), (14), and (16) we get

$$\begin{aligned}\phi_1(\|\underline{x}(t)\|)&\le v_1(t,\underline{x}(t))\le a\|\underline{V}(t_0,\underline{x}_0)\|\exp[-b(t-t_0)]\\&\le a\psi_{t_0}(\|\underline{x}_0\|)\exp[-b(t-t_0)]\quad\forall t\ge t_0\end{aligned} \tag{17}$$

If we choose $\|\underline{x}_0\|<c$ sufficiently small, then all the inequalities in (17) are preserved; moreover we get

$$\begin{aligned}\|\underline{x}(t)\|&\le\phi_1^{-1}\{a\psi_{t_0}(\|\underline{x}_0\|)\exp[-b(t-t_0)]\}\\&\triangleq\beta_{t_0}(\|\underline{x}_0\|,t-t_0)\quad\forall t\ge t_0\ \forall\|\underline{x}_0\|<c\end{aligned} \tag{18}$$

Note that $\phi_1^{-1}\in K$ since $\phi_1\in K$ [2]. Moreover $\beta_{t_0}\in KL$, then using Corollary 1 implies AS of ZES of (1).

Part b) If $\underline{V}(t,\underline{x})$ is decrescent, then using ψ instead of ψ_{t_0}, all the above relationships used in proof of part (*a*), specially (18) hold independent of t_0. Using Lemma 2 implies the UAS of ZES.

Part c) Similarly to the proof of part (*a*) $\underline{V}(t_0,\underline{0})=\underline{0}$ and $\underline{V}(t_0,\underline{x}_0)$ is continuous everywhere, thus applying the Lemma 1(iii), (14) holds for t_0 globally. Moreover by the PDF and RU assumptions for $v_1(t,\underline{x})$ the condition (13) is satisfied globally for $\phi_1\in K_\infty$. Using (13) and (14) globally implies (17) and (18) are satisfied globally, thus ZES is GAS using Corollary 1.

Part d) Similarly to the proof of part (*c*), the relationship (18) holds globally, but for ψ instead of ψ_{t_0}. So the ZES of (1) is UGAS using Lemma 2. ■

The above theorem relates the stability of ZES of a given nonlinear system (1) to the stability of LTI co-system (11), which is too restrictive. Of course this theorem is introduced to find the sufficient conditions of our main theorem. Thus consider the following differential inequality

$$v_1^{(m)}(t,\underline{x})+a_{m-1}v_1^{(m-1)}(t,\underline{x})+\cdots+a_0v_1(t,\underline{x})=d_m(t,\underline{x})\le 0 \tag{19}$$

The relationship (19) is restricting, because: First, a linear combination of a LF and its higher order derivatives must be negative, but in each derivation some new terms may be included in the LF derivatives, so it might be hard to determine the sign of a linear combination of them. Second, the LF derivatives might be non-smooth; in this case the derivation procedure (3) cannot be iterated.

To obtain a solution, considering (19) as an LTI co-system with a non-positive input, the controllable canonical form state space equation of this system is presented as follows:

$$\begin{bmatrix}\dot{v}_1(t,\underline{x})\\\dot{v}_2(t,\underline{x})\\\vdots\\\dot{v}_m(t,\underline{x})\end{bmatrix}=\begin{bmatrix}0&1&\cdots&0\\0&0&\ddots&\vdots\\0&0&\cdots&1\\-a_0&-a_1&\cdots&-a_{m-1}\end{bmatrix}\begin{bmatrix}v_1(t,\underline{x})\\v_2(t,\underline{x})\\\vdots\\v_m(t,\underline{x})\end{bmatrix}+\begin{bmatrix}d_1(t,\underline{x})\\d_2(t,\underline{x})\\\vdots\\d_m(t,\underline{x})\end{bmatrix} \tag{20}$$

where $d_i(t,\underline{x})=0$ for $i=1,\dots,m-1$. For more freedom, the state space equation (20) is considered with $d_i(t,\underline{x})\le 0$ for $i=1,\dots,m$. Thus the co-system (20) can be written in the form of following inequalities.

$$\begin{cases}\dot{v}_i(t,\underline{x})\le v_{i+1}(t,\underline{x}) & ,\quad i=1,2,\dots,(m-1)\\\dot{v}_m(t,\underline{x})\le-\sum_{i=1}^m a_{i-1}v_i(t,\underline{x})\end{cases} \tag{21}$$

Let us explain our approach: Consider a C^1 LF candidate $v_1(t,\underline{x})$, if $\dot{v}_1(t,\underline{x})$ is not negative definite, so by using this LF we may not conclude the stability of ZES of (1) using the Lyapunov direct method. In this paper, the higher order time derivatives, $v_i(t,\underline{x}) = v_1^{(i-1)}(t,\underline{x}),\ i = 2,3,\ldots,m$, will be used in the stability analysis, to compensate the nonnegative definiteness of $\dot{v}_1(t,\underline{x})$. However, if the nonlinear system (1) and/or the function $v_1(t,\underline{x})$ are not smooth enough, then the higher order derivatives $v_1^{(i-1)}(t,\underline{x})$ are not well defined. In this case, we propose the C^1 functions $v_i(t,\underline{x})$ to be found iteratively instead of $v_1^{(i-1)}(t,\underline{x})$ as follows: compute $\dot{v}_i(t,\underline{x})$, if it is not smooth enough, then choose an upper bounding C^1 function $v_{i+1}(t,\underline{x}) \ge \dot{v}_i(t,\underline{x})$ (The equality is permitted for the smooth cases). Repeat this work for $i = 1,2,\ldots,m-1$, until the last inequality in (21) for some m is satisfied.

4. THE MAIN RESULTS

The main theorem of this paper is the following:

Theorem 2: Consider the nonlinear system (1) and a C^1 m-vector function $\underline{V}(t,\underline{x})$ of the form (2) whose derivative $\dot{\underline{V}}(t,\underline{x})$ along the solutions of (1) satisfies the controllable canonical form (20). Moreover suppose that:

i. The inputs of (20) have non positive values, i.e.,

$$d_i(t,\underline{x}) \le 0 \qquad , \quad i = 1,2,\ldots,m \tag{22}$$

ii. All the poles of the LTI co-system (20) (roots of the characteristic eq. (7)) are negative real ones.

Therefore:

a) If $\underline{V}(t,\underline{0}) = \underline{0}$ and $v_1(t,\underline{x})$ is LPDF, then the ZES of (1) is AS.

b) If $\underline{V}(t,\underline{x})$ is decrescent, while its first component $v_1(t,\underline{x})$ is LPDF, then the ZES of (1) is UAS.

c) If $\underline{V}(t,\underline{0}) = \underline{0}$ and $v_1(t,\underline{x})$ is RU and PDF, then the ZES of (1) is GAS.

d) If $\underline{V}(t,\underline{x})$ is globally decrescent, while its first component $v_1(t,\underline{x})$ is RU and PDF, then the ZES of (1) is UGAS.

e) If $v_1(t,\underline{x})$ is PDF then – with no further assumptions - the ZES of (1) is globally attracting. ■

We need some preliminaries on conditions (i) and (ii) of Theorem 2, before proving this theorem. Example 1 shows the difficulties arisen by complex conjugated poles of the characteristic equation (7), thus the assumption (ii) is to avoid this case. Considering a given trajectory $\underline{x}(t,t_0,\underline{x}_0)$, each of $v_i(t,\underline{x})$ and $d_i(t,\underline{x})$ functions in (20) is a function of *t*. In the following $v_1(t,\underline{x}) = y(t)$ is considered as an output to (20), and by the superposition rule for the output of (20) one has:

$$v_1(t,\underline{x}(t)) \triangleq y(t) = y_0(t) + y_d(t) \tag{23}$$

where $y_0(t)$ is the response to the initial conditions $v_i(t_0,x_0)$ and $y_d(t)$ is the response to the negative inputs $d_i(t_0,x_0)$. Using the negative real eigenvalues, the zero input response of (20) is AS i.e., $y_0(t) \to 0$. But we require $y(t) \to 0$. To compare the outputs $y(t)$ and $y_0(t)$, we only need to concentrate on $y_d(t)$, i.e., notice the transfer functions in the following lemma:

Lemma 3: Consider the controllable canonical form system (20) with the output (23). The transfer function $h_i^m(s)$ from each input $D_i(s)$ to the output $Y(s)$ is given as:

$$h_i^m(s) = \frac{a_i + a_{i+1}s + a_{i+2}s^2 + \cdots + a_{m-1}s^{m-1-i} + s^{m-i}}{a_0 + a_1 s + a_2 s^2 + \cdots + a_{m-1}s^{m-1} + s^m} \triangleq \frac{\Delta_i^m}{\Delta_0^m} \tag{24}$$

Proof: We know that

$$h_i^m(s) = \Delta_i^m / \Delta_0^m = \left[1\,0\cdots0\right](sI-A)^{-1}\left[0\cdots1\cdots0\right]^T =$$

$$\frac{1}{\det(sI-A)}\left[1\,0\cdots0\right]A^*(s)^T\left[0\cdots1\cdots0\right]^T = \frac{a_{i1}^*(s)}{\det(sI-A)}$$

where $A^*(s) = [a_{ij}^*(s)]_{m\times m}$ is the cofactor matrix of $(sI-A)$. We only need to compute the first column of $A^*(s)$, i.e., all the $a_{i1}^*(s)$ components, by deleting the *i*'th row and the first column of $(sI-A)$ and reaching a $(m-1)\times(m-1)$ determinant and multiplying by $(-1)^{i+1}$. This determinant is expanded on the last row, so we get to the following description: (The dashed line shows the place of deleted row)

$$\Delta_i^m = a_{i1}^*(s) = (-1)^{i+1}\begin{vmatrix} -1 & & & & & & \\ s & -1 & & & & & \\ & s & -1 & & & & \\ \hdashline & & & s & -1 & & \\ & & & & s & -1 & \\ & & & & & s & -1 \\ a_1 & a_2 & \cdots & a_i & \cdots & a_{m-2} & s+a_{m-1} \end{vmatrix} \; (i-1) =$$

$$(-1)^{i+1}\left\{(-1)^m a_1 \underbrace{\begin{vmatrix} 0 & & & & \\ -1 & 0 & & & \\ & s & \ddots & & \\ & & \ddots & -1 & \\ & & & s & -1 \end{vmatrix}}_{0} + (-1)^{m+1} a_2 \underbrace{\begin{vmatrix} -1 & & & & \\ s & 0 & & & \\ & s & \ddots & & \\ & & \ddots & -1 & \\ & & & s & -1 \end{vmatrix}}_{0}\right.$$

$$+ \cdots + a_{i-1} \times 0 +$$

$$(-1)^{m+i-1}a_i\begin{vmatrix}-1 & & & & \\ s & -1 & & & \\ & s & \ddots & & \\ & & \ddots & -1 & \\ & & & s & -1\end{vmatrix}+(-1)^{m+i}a_{i+1}\begin{vmatrix}-1 & & & & \\ s & -1 & & & \\ & & s & & \\ & & & -1 & \\ & & & s & -1\end{vmatrix}$$

$$+\cdots+(-1)^{2m-3}a_{m-2}\times$$

$$\begin{vmatrix}-1 & & & & & \\ s & -1 & \overbrace{}^{(m-i-2)} & & & \\ & & s & -1 & & \\ & & & s & -1 & \\ & & & & s & -1\end{vmatrix}+(-1)^{2m-2}(s+a_{m-1})\begin{vmatrix}-1 & & & & \\ s & -1 & \overbrace{}^{(m-i-1)} & & \\ & & s & -1 & \\ & & & s & -1 \\ & & & & s\end{vmatrix}$$

$$\Rightarrow \Delta_i^m = a_{i1}^*(s) = a_i + a_{i+1}s + a_{i+2}s^2 + \cdots + a_{m-1}s^{m-1-i} + a_m s^{m-i}$$

(25)

Definition 1 [14]: A linear system is said to be *externally positive* if for any positive input, the system's output is also positive. It is said *positive*, if all the states remain positive as well.

Lemma 4: If poles of the controllable canonical form system (20) all have *negative real* values, then the system is externally positive, considering the inputs and the output of Lemma 3.

Proof: To prove this lemma we use the following facts: First, An LTI system with a transfer function $h(s)$ is externally positive iff $h(t) > 0$ for all $t > 0$. Second, the summation or the multiplication of any collection of externally positive LTI systems is also an externally positive system.

We prove the externally positive-ness of $h_i^m(s)$ transfer functions for $i = 1, 2, \ldots, m$ by *mathematical induction* on $m \in \mathbb{N}$. Note that $h_1^1(s) = 1/(a_0 + s)$, i.e., $h_1^1(t) = e^{-a_0 s}\mathbf{1}(t)$ is externally positive. Thus the claim of lemma is true in the case of dimension $m = 1$. Let this claim for $m \in \mathbb{N}$ be true. We will show it for $m+1$. Suppose the denominator of $h_i^{m+1}(s)$ transfer functions is given by:

$$\Delta_0^{m+1} = (a+s)(a_0 + a_1 s + a_2 s^2 + \cdots + a_{m-1}s^{m-1} + s^m) \qquad (26)$$

where $(-a)$ and all other poles are negative real. Thus $h_{m+1}^{m+1}(s) = 1/(a+s)(a_0 + a_1 s \cdots + a_{m-1}s^{m-1} + s^m)$ is the product of $m+1$ externally positive first order systems and hence it is externally positive. The relationship (26) could be written in the following form

$$\Delta_0^{m+1} = aa_0 + (aa_1 + a_0)s + (aa_2 + a_1)s^2 + \cdots + (aa_{m-1} + a_{m-2})s^{m-1}$$

$$+(a + a_{m-1})s^m + s^{m+1} \qquad (27)$$

Using this form and comparing with (24), the numerator of $h_i^{m+1}(s)$ with some rearranging of terms, is given by:

$$\Delta_i^{m+1} = (aa_i + a_{i-1}) + (aa_{i+1} + a_i)s + \cdots + (aa_{m-1} + a_{m-2})s^{m-1-i}$$

$$+(a + a_{m-1})s^{m-i} + s^{m+1-i}$$

$$\Delta_i^{m+1} = a_{i-1} + (a+s)(a_i + a_{i+1}s + \cdots + a_{m-1}s^{m-1-i} + s^{m-i}) \qquad (28)$$

Then dividing (28) by (27) and using (24) yield:

$$h_i^{m+1}(s) = \frac{\Delta_i^{m+1}}{\Delta_0^{m+1}} = \frac{a_{i-1}}{\Delta_0^{m+1}} + \frac{\Delta_i^m}{\Delta_0^m} = a_{i-1}h_{m+1}^{m+1}(s) + h_i^m(s) \qquad (29)$$

Where $h_{m+1}^{m+1}(s)$ and $h_i^m(s)$ are both externally positive transfer functions of degrees $m+1$ and m respectively, thus $h_i^{m+1}(s)$ is also externally positive.

Remark 1: The statement "any LTI system with negative real poles is externally positive" is wrong. The transfer function $h(s) = 1/(s+1) - 2/(s+2) + 1/(s+5)$ is a counterexample.

Remark 2: The co-system (20) is not internally positive.

Proof of Theorem 2: By the Lemma 4 the co-system (20) is an externally positive system, while it is linear with negative inputs $d_i(t, \underline{x})$, thus $y_d(t) \le 0$ in (23). Using this relationship implies $v_1(t, \underline{x}) = y(t) \le y_0(t)$, where $y_0(t)$ is the exponentially stable and depends only on $\underline{\mathrm{V}}(t_0, \underline{x}_0)$, thus there are $a, b > 0$ constants such that:

$$v_1(t, \underline{x}) \le y_0(t) \le |y_0(t)| \le a\left\|\underline{\mathrm{V}}(t_0, \underline{x}_0)\right\|\exp[-b(t - t_0)] \qquad (30)$$

The above relationship is like that of (16). Thus this theorem could be proven with a same manner of Theorem 1. Especially, the proof of each part (*a*)-(*d*) in Theorem 1 will be used for the same part of Theorem 2, respectively. We only prove the part (*e*) of Theorem 2.

Part e) The equation (30) is satisfied globally, thus $\lim_{t\to\infty} v_1(t, \underline{x}(t, t_0, \underline{x}_0)) = 0$. Using (13), implies the globally attraction of ZES. ■

5. THE EXAMPLES

Here some examples are given in order to show the effective-ness of the approach. First an example is presented to show that using the Theorem 1 with the normal quadratic PDF functions gives a necessary and sufficient condition for the LTI systems stability. Another example shows a similar result in the case of nonlinear homogeneous polynomial systems of zero degree of homogeneity [15].

Example 2: Consider the following LTI system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -4 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad (31)$$

The eigenvalues of this system are $s_{1,2}=-1\pm 2i$, thus the system is AS using the second method of Lyapunov. Now consider the following PDF function:

$$\begin{aligned}v(\underline{x})&=x_1^2+x_2^2\\ \dot{v}(\underline{x})&=2x_1\dot{x}_1+2x_2\dot{x}_2=-2x_1^2-6x_1x_2-2x_2^2\end{aligned} \qquad (32)$$

$\dot{v}(\underline{x})$ is not negative definite, since $\Delta=(-6)^2-4(-2)(-2)=20>0$. So the Lyapunov direct method fails using this LF candidate. The $\dot{v}(\underline{x})$ function and any of the higher order LF derivatives are also in quadratic forms. So we may find a linear dependence between them. The computation yields:

$$\begin{aligned}\ddot{v}(\underline{x})&=28x_1^2+24x_1x_2-2x_2^2\\ \dddot{v}(\underline{x})&=-152x_1^2+24x_1x_2+28x_2^2\end{aligned} \qquad (33)$$

Using the bases $\{x_1^2\ ,x_1x_2\ ,x_2^2\}$, one has:

$$[v\ \ \dot{v}\ \ \ddot{v}\ \ \dddot{v}]=[x_1^2\ \ x_1x_2\ \ x_2^2]\begin{bmatrix}1 & -2 & 28 & -152\\ 0 & -6 & 24 & 24\\ 1 & -2 & -2 & 28\end{bmatrix} \qquad (34)$$

Let us find a linear combination of $v^{(i)}$'s as follows

$$r_0v+r_1\dot{v}+r_2\ddot{v}+\dddot{v}=[v\ \ \dot{v}\ \ \ddot{v}\ \ \dddot{v}][r_0\ r_1\ r_2\ 1]^T=0 \qquad (35)$$

Substituting (34) for (35) implies:

$$\begin{bmatrix}1 & -2 & 28 & -152\\ 0 & -6 & 24 & 24\\ 1 & -2 & -2 & 28\end{bmatrix}\begin{bmatrix}r_0\\ r_1\\ r_2\\ 1\end{bmatrix}=0\ or\ \begin{bmatrix}1 & -2 & 28\\ 0 & -6 & 24\\ 1 & -2 & -2\end{bmatrix}\begin{bmatrix}r_0\\ r_1\\ r_2\end{bmatrix}=\begin{bmatrix}152\\ -24\\ -28\end{bmatrix}$$

Thus solving for r_0, r_1 and r_2 yields:

$$40v+28\dot{v}+6\ddot{v}+\dddot{v}=0 \qquad (36)$$

The characteristic equation of (36), i.e.,

$$\Delta(s)=s^3+6s^2+28s+40=(s+2)[(s+2)^2+16]=0 \qquad (37)$$

is a Hurwitz polynomial. Using Theorem 1 implies ZES of (31) is GAS, which coincides with the eigenvalue analysis given above. The roots of $\Delta(s)$ are sum of the eigenvalues of the first LTI system, namely $s_{1,2}=-1\pm 2i$. This property is preserved for all LTI systems. ■

The next example, introduces a generalized homogeneous nonlinear system with zero degree of homogeneity (see [15]).

Example 3: Consider the following nonlinear differential equation, with unknown real parameters *a*, *b*, and *c*.

$$\begin{cases}\dot{x}_1=ax_1\\ \dot{x}_2=bx_1^2+cx_2\end{cases} \qquad (38)$$

The linearization of this system in the neighborhood of the origin i.e. $\{\dot{x}_1=ax_1\ \ ,\dot{x}_2=cx_2\}$ represents an AS system iff $a,c<0$. So the ZES of the nonlinear system (38) is AS if $a,c<0$. We prove that $a,c<0$ is necessary and sufficient for the GAS of ZES as well.

The independence of the stability condition of the parameter *b*, mast not confuse us, because the nonlinear system (38) is in a lower triangular form and it can be considered as a cascade combination of the two linear systems $\{\dot{x}_1=ax_1\}$ and $\{\dot{x}_2=cx_2+u\}$ with $u=bx_1^2$.

The nonlinear system (38) is in a form that the x_1^2 and x_2 can be viewed as interchanged terms to construct a LF candidate $v(\underline{x})=(x_1^2)^2+(x_2)^2$. Computation yields:

$$\begin{aligned}\dot{v}(\underline{x})&=4x_1^3\dot{x}_1+2x_2\dot{x}_2=4ax_1^4+2bx_1^2x_2+2cx_2^2\\ \ddot{v}(\underline{x})&=(16a^2+2b^2)x_1^4+(4ab+6bc)x_1^2x_2+(4c^2)x_2^2\\ \dddot{v}(\underline{x})&=(64a^3+12ab^2+6cb^2)x_1^4+(8a^2b+16abc+14bc^2)x_1^2x_2\\ &+(8c^3)x_2^2\end{aligned}$$

The $v(\underline{x})$, $\dot{v}(\underline{x})$, $\ddot{v}(\underline{x})$ and $\dddot{v}(\underline{x})$ functions (and all other derivatives) are linear combinations of $\{x_1^4\ ,x_1^2x_2\ ,x_2^2\}$. Thus one has:

$$[v\ \ \dot{v}\ \ \ddot{v}\ \ \dddot{v}]=$$
$$[x_1^4\ \ x_1^2x_2\ \ x_2^2]\begin{bmatrix}1 & 4a & 16a^2+2b^2 & 64a^3+12ab^2+6cb^2\\ 0 & 2b & 4ab+6bc & 8a^2b+16abc+14bc^2\\ 1 & 2c & 4c^2 & 8c^3\end{bmatrix}$$

A linear dependence of these functions can be found by solving (35) for r_0,r_1,r_2 or the following equation:

$$\begin{bmatrix}1 & 4a & 16a^2+2b^2 & 64a^3+12ab^2+6cb^2\\ 0 & 2b & 4ab+6bc & 8a^2b+16abc+14bc^2\\ 1 & 2c & 4c^2 & 8c^3\end{bmatrix}\begin{bmatrix}r_0\\ r_1\\ r_2\\ 1\end{bmatrix}=0 \qquad (39)$$

Solving (39) by the Gauss method yields:

$$\begin{cases}r_0=-(16a^2c+8ac^2)\quad ,\quad r_1=8a^2+16ac+2c^2\\ r_2=-(6a+3c)\end{cases}$$

So the following linear dependence is obtained:

$$-(16a^2c+8ac^2)v+(8a^2+16ac+2c^2)\dot{v}-(6a+3c)\ddot{v}+\dddot{v}=0$$

The new characteristic polynomial is given as:

$$\begin{aligned}\Delta(s)&=s^3-(6a+3c)s^2+(8a^2+16ac+2c^2)s-(16a^2c+8ac^2)\\ &=(s-2c)(s-4a)[s-(2a+c)]\end{aligned} \qquad (40)$$

Thus the sufficient conditions for GAS of ZES are given:

$$c<0\ ,a<0 \qquad (41)$$

The result coincides with what was found earlier. ■

Example 4: A nonlinear time varying system is given

$$\begin{cases} \dot{x} = -g_1(x) - 2xy + ke^{-pt}x \\ \dot{y} = -g_2(y) + x^2 + ke^{-pt}y \end{cases} \tag{42}$$

$$\sigma^2 \le g_i(\sigma)\sigma \le 2\sigma^2, \forall i = 1,2 \tag{43}$$

Considering non-smooth functions $g_i(\sigma)$, the UGAS of ZES of this system will be shown when

$$p > 2k > 2 \tag{44}$$

Using a PDF $v_1 = x^2 + 2y^2$, computing $\dot{v}_1$ and eliminating the $g_i(\cdot)$ terms implies:

$$\dot{v}_1 = 2x\dot{x} + 4y\dot{y} = -2xg_1(x) - 4yg_2(y) + (x^2 + 2y^2)2ke^{-pt} \\ \le -2(x^2 + 2y^2) + (x^2 + 2y^2)2ke^{-pt} = 2v_1(ke^{-pt} - 1) \triangleq v_2 \tag{45}$$

Clearly $\dot{v}_1$ is not negative definite. Replace $t = 0$ in (45) and notice $k - 1 > 0$ in (44) for the reason. Thus the Lyapunov direct method fails using this function. However, $\dot{v}_1$ approaches to a negative definite function. The $\dot{v}_1$ function is not smooth, but the v_2 function in (45) is smooth. Computing $\dot{v}_2$, replacing $\dot{v}_1$ from (45), and rearranging terms yields

$$\dot{v}_2 = -p2v_1ke^{-pt} + 2\dot{v}_1(ke^{-pt} - 1) = -p2v_1ke^{-pt} + \\ 2[-2xg_1(x) - 4yg_2(y) + 2v_1ke^{-pt}](ke^{-pt} - 1) = \\ -(p+2)2v_1ke^{-pt} + 4v_1(ke^{-pt})^2 \\ +4[xg_1(x) + 2yg_2(y)] - 4[xg_1(x) + 2yg_2(y)]ke^{-pt} \tag{46}$$

Then using the bounds of (43) and eliminating the $g_i(\sigma)$ functions from $\dot{v}_2$, one obtains

$$\dot{v}_2 \le -(p+2)2v_1ke^{-pt} + 4v_1(ke^{-pt})^2 + \\ 8[x^2 + 2y^2] - 4[x^2 + 2y^2]ke^{-pt} = \\ -(p+4)2v_1ke^{-pt} + 4v_1(ke^{-pt})^2 + 8v_1 \tag{47}$$

Then using (45) and (47) arrange the following linear combination:

$$\dot{v}_2 + a_1v_2 + a_0v_1 \le -(p+4)2v_1ke^{-pt} + 4v_1(ke^{-pt})^2 + 8v_1 \\ +a_1 2v_1(ke^{-pt} - 1) + a_0v_1 = \\ = 2v_1ke^{-pt}[2ke^{-pt} - (p+4) + a_1] + v_1[8 + a_0 - 2a_1] \\ \le 2v_1ke^{-pt}\underbrace{[2k - p - 4 + a_1]}_{=0} + v_1\underbrace{[8 + a_0 - 2a_1]}_{=0} \tag{48}$$

Define a new variable $b \triangleq p - 2k > 0$ (see (44)), then set the right-hand side brackets of (48) equal to zero, as follows

$$\begin{cases} [2k - p - 4 + a_1] = -b - 4 + a_1 = 0 \\ 8 + a_0 - 2a_1 = 0 \end{cases} \tag{49}$$

Solving (49) yields $a_1 = b + 4$ and $a_0 = 2b$. Using (45) and (48) and the quantities for a_0 and a_1, one has

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} \le \begin{bmatrix} 0 & 1 \\ -2b & -(b+4) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{50}$$

So $v_1(t, \underline{x})$ and $v_2(t, \underline{x})$ satisfy (21). The characteristic equation

$$\Delta(s) = s^2 + (b+4)s + 2b = 0 \tag{51}$$

is Hurwitz and has negative real zeros, because $\Delta_1 = (b+4)^2 - 4(1)(2b) = b^2 + 16 > 0$. So $\underline{V}(t, \underline{x})$ satisfies the conditions of part (*d*) of Theorem 2, because the first component $v_1(t, \underline{x})$ is RU and PDF and both components $v_1(t, \underline{x}), v_2(t, \underline{x})$ are globally decrescent, hence the ZES of (42) is UGAS. ■

The last inequality (50) in Example 4 was found analytically. This is not a possible way for all nonlinear systems, and a question is: how one can find generally such inequality? An answer is as follows: for a given $\underline{x}_k \in \mathbb{R}^n$ and a given $t_k \in \mathbb{R}$ the last inequality in (21) is considered as a linear inequality $\dot{v}_m(t_k, \underline{x}_k) + \sum_{i=1}^{m} a_{i-1}v_i(t_k, \underline{x}_k) \le 0$ for the positive weights $a_i > 0$. An arbitrary set of such points produces a linear matrix inequality (LMI) for the $a_i > 0$. Such LMIs have been solved for the nonlinear homogeneous systems in [16]-[17], where the radial symmetry of homogeneous systems reduced the size of LMIs considerably. After finding the a_i coefficients if the characteristic equation of (20) does not have only negative real roots then the above procedure must be tried again to obtain suitable functions.

6. CONCLUSION

In this paper, we replaced the negative definiteness of $\dot{v}(t, \underline{x})$ in the Lyapunov stability theory with some inequalities for the higher order derivatives $v^{(i)}(t, \underline{x})$ of the LF candidate. The inequalities are presented and are solved using a new controllable canonical LTI system. If the new LTI system has negative real poles, then AS of ZES for the nonlinear system is obtained. Some examples are given to show the approach. All theorems state sufficient conditions for the stability of nonlinear systems and the converses of them are not true, i.e. when the characteristic equation is not Hurwitz we cannot conclude the instability of the nonlinear systems.

7. APPENDIX. THE COMPARISON PRINCIPLE

The well-known comparison principle in the nonlinear analysis was first introduced for scalar differential inequalities [2]. The generalization of this principle to a vector form is as follows:

Lemma 5 [7]: (Generalized Comparison Lemma) Consider a vector equation $\underline{\dot{u}} = \underline{g}(t,\underline{u}), \quad \underline{u}(t_0) = \underline{u}_0$, $\underline{u} \in \mathbb{R}^m$, where $\underline{g}(t,\underline{u})$ is continuous in *t* and locally Lipschitz in $\underline{u}$. Let $\underline{v}(t)$ be a continuous vector function whose upper right-hand derivative $D^+\underline{v}(t)$ satisfies the following differential inequality component-vise:

$$D^+\underline{v}(t) \le \underline{g}(t,\underline{v}(t)), \quad \underline{v}(t_0) \le \underline{u}_0 \qquad (52)$$

If $\underline{g}(t,\underline{u})$ is of class W, i.e. for each $i = 1,\dots,m$, $g_i(t,\underline{a}) \le g_i(t,\underline{b})$ wherever $a_i = b_i$ and $\forall j = 1,\dots,m$, $j \ne i$, $a_j \le b_j$ then $\underline{v}(t) \le \underline{u}(t)$ for $t \ge t_0$.

■

It is clear that every scalar function $g(t,u)$ is of class W, and an LTI $\underline{g}(\underline{u}) = A\underline{u}$ is of class W iff all off-diagonal elements of A are positive.

8. NOMENCLATURE

ZES	Zero Equilibrium State
LF	Lyapunov Function
(L)PDF	(Locally) Positive Definite Function
RU	Radially Unbounded
(U)(G)(A)S	(Uniformly) (Globally) (Asymptotically) Stable
VLF	Vector Lyapunov Function
LTI	Linear Time Invariant
$\|\cdot\|$	A given norm on $\mathbb{R}^n$
$B_r = \{\underline{x} : \|\underline{x}\| < r\}$	Open ball of radius *r*
$\underline{x}(t,t_0,\underline{x}_0)$	A trajectory $\underline{x}(t)$ starting at $\underline{x}(t_0) = \underline{x}_0$
$\underline{u}$	The underline means a vector quantity
$\underline{V} : \mathbb{R}\times\mathbb{R}^n \rightarrow \mathbb{R}^m$	Vector function of dimension m
$\alpha \in K(K_\infty)$	α is a function of class K (K infinity) [2]
$\beta \in KL$	β is a function of class KL [2]
$v^{(i)}(t,\underline{x})$	The *i*'th total time derivative of $v(t,\underline{x})$ along the solutions of (1) ($\ne$ partial deriv.)

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