Near Pole Polar Diagram of Points and its Duality with Applications

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ABSTRACT

In this paper we propose a new approach to plane partitioning with similar features to those of Polar Diagram, but we assume that the pole is close to the sites. The result is a new tessellation of the plane in regions called Near Pole Polar Diagram NPPD. Here we define the (NPPD) of points, the dual and the Contracted dual of it, present an optimal algorithms to draw them and discuss the applications and optimality of the algorithms.

KEYWORDS

Polar Diagram, Near Pole Polar Diagram, NPPD, Voronoi Diagram, Computational Geometry, Graph Theory.

1. INTRODUCTION

One of the most fundamental concepts in Computational Geometry is the Voronoi diagram. Its algorithms and applications have been studied extensively [10, 9, 6]. This concept has also been generalized in a variety of directions by replacing the Euclidean distance with other metrics such as L_p-distance [16], weighted distances [13, 15], the geodesic distance[11, 4], the power distance[12, 14], and a skew distance[7]. However, some of them are difficult to compute. As the solution to many problems in computational geometry requires some kind of angle processing of the input, some other generalizations of Voronoi diagram based on angle have been studied in [2, 3]. Grima et al. propose a new locus approach for problems processing angles, the Polar Diagram. For any position q in the plane (represented by a point) the site with smallest polar angle, is the owner of the region where q lies into. Grima et al. [2] proved that polar diagram, used as preprocessing, can be applied to many problems in computational geometry in order to speed up their processing times. Some of these applications are the Convex Hull, Visibility problems, and Path Planning problems. Jarvis's March approach can be improved to become an optimal time process and Visibility problems can take advantage of polar diagram principles as well. Also we recently introduced in [1, 3] the dual of polar diagram, the polar diagram with respect to a near pole for a set of points and some properties and applications. In this paper we introduce the NPPD for a set of points and define its dual. Then we present an optimal algorithm to draw the contracted dual of NPPD. In the following we are going to review some definitions and properties that we need in the next sections.

Polar Diagram is the plane partition with similar features to those of the Voronoi Diagram. In fact, the Polar Diagram can be seen in the context of the generalized Voronoi Diagram. The Polar Angle of the point p with respect to s_i, denoted as $ang_{s_i}(p)$, is the angle formed by the positive horizontal line of p and the straight line linking p and s_i (see figure1).

Given a set S of n points on the plane E^2, the locus of points having smaller positive polar angle with respect to $s_i \in S$ is called Polar Region of s_i. Thus,

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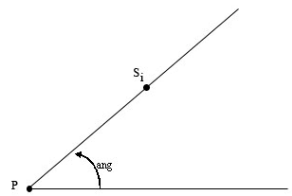

Figure 1: polar angle

$$P_S(s_i) = \{(x, y) \in E^2 \mid ang_{s_i}(x, y) < ang_{s_j}(x, y); \forall j \neq i\}$$

The plane is divided into different regions in such a way that if the point $(x, y) \in E^2$ lies into $P_S(s_i)$, it is known that s_i is the first site found performing an angular scanning starting from (x, y). We can draw an analogy between this angular sweep and the behavior of a radar [8]. Figure 2 depicts the polar diagram of a set of points on the plane. In the polar diagram with respect to a near pole, it is assumed that the pole is located close to the sites. Although the polar diagram of n points on the plane is not a graph, we define its dual as a dual of graph. If we omit the loops and replace the parallel edges with one edge in the dual of polar diagram, then we will have another graph named Contracted Dual of Polar Diagram (CDPD) (Figure 3). There is an optimal algorithm to find the CDPD in [3].

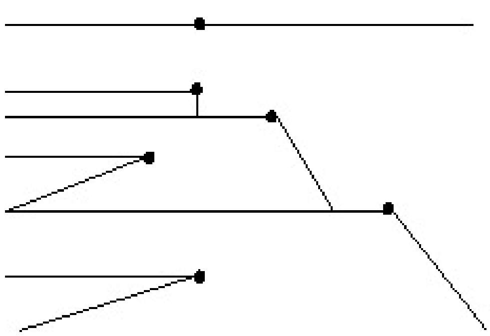

Figure 2: polar diagram of 6 points

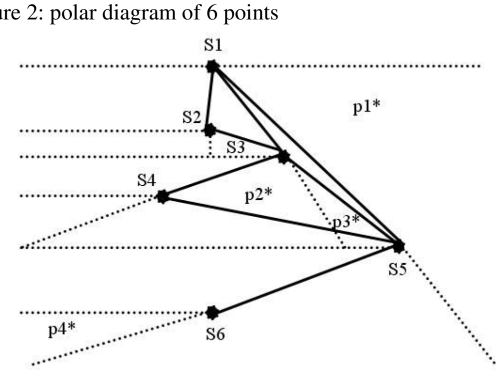

Figure 3: Contracted dual of polar diagram

This paper is structured as follows: In the next section we introduce the NPPD and present our optimal incremental approach for this problem. Then we introduce some applications in subsection 2.3. In section 3 the contracted dual of NPPD is introduced and we propose our optimal algorithm to draw it. Section 4 contains the conclusion and future works.

2. NEAR POLE POLAR DIAGRAM FOR A SET OF POINTS

In this section we introduce the Near Pole Polar Diagram of a set of n point sites on the plane. Also we are going to present an optimal algorithm to draw it. Our approach is an incremental approach.

2.1 Near Pole Polar Diagram (NPPD)

We introduce the Near Pole Polar Diagram with similar features to those of the Polar diagram that can be seen in the context of the generalized Voronoi diagram [6]. As described in [3, 2], the pole mentioned lies on the left hand side of the plane at $-\infty$. In the polar diagram with respect to a near pole, it is assumed that the pole is located on the left hand side of the sites close to them. This allows us to find more applications for the problem. For example, the pole can be considered as the center of vision (eye) of a robot.

In addition to the given point sites, the point p in the plane is also given as a pole, and the partitioning of the plane will depend on the position of p. Without loss of generality assume that the pole p is located on the left-hand side of the sites. Figure 4 shows an example of NPPD for 7 points with respect to pole p.

In short, a near pole polar diagram can be described as follows. Initially there is a radar at each of the point sites looking at the pole. They simultaneously start to rotate in counterclockwise direction and scan their periphery. The region in the plane observed by radar p_i before other radars will be called the region of p_i's NPPD and is denoted by $NPPD_S(p_i)$. Also we denote the NPPD of the set of point sites S with respect to pole p by $NPPD\ (S, p\ )$.

In Figure 5, we are given sites s_1 and s_2, pole p and point x in the plane. Since $ps_1x < ps_2x$, in the plane's partition, x will belong to the region of s_1. However this partitioning will produce disconnected regions with curved boundaries and makes the problem more complex and drawing the corresponding diagram more difficult. In this paper we have made an assumption which not only makes the problem simpler, but also increases its applications.

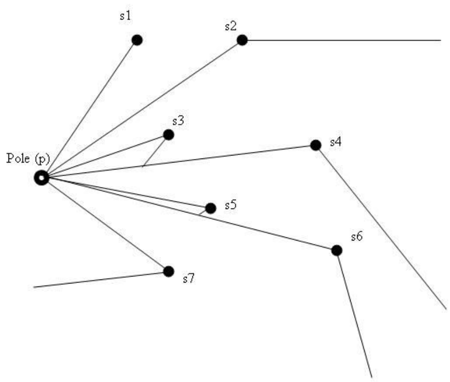

Figure 4: Near pole polar diagram of 7 points

Assumption: The line of vision of each site blocks that of any other site that intersects it later.

2.2 Incremental approach

In this subsection, we are going to present an incremental algorithm for drawing NPPD, working in optimal time. In this paradigm, we first compute the tangent of the line segments $s_i p; i = 1, 2, ..., n$ and then we sort them. Then a straight half line starting at pole p rotates in clockwise direction around the pole p and sweeps the plane. The NPPD region of s_i is built according to the following theorem.

Theorem 2.1. Let S'_i denote the set of processed points when point s_i is reached, $S'_i = S'_{i-1} \cup s_i$. If $s_i \in NPPD_S(s_k), s_k \in S'_{i-1}$ then the near pole polar region of s_i is the angular sector defined by the half line from s_i to the pole p and the half line defined by s_i and s_k, which does not contain s_k.

Proof. Consider point x within the region mentioned in the theorem. Since according to our initial assumption the line segment ps_i blocks other sites' (and specially s_k's) line of view, then the above mentioned region in NPPD belongs to s_i. Also as $s_k py < s_i py$, there can not exist any other points such as y within s_i's region (see Figure 6).

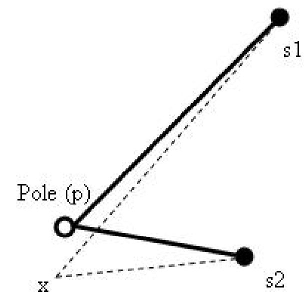

Figure 5: Our assumption: the line segment ps_2 blocks s_1 's line of view.

Theorem 2.1 is the key to compute the NPPD using the Incremental method. Algorithm 1 describes the process: $S = \{s_0, s_1, ..., s_{n-1})$ is given. $TS = \{ts_0, ts_1, ..., ts_{n-1})$ is sorted from the largest member to the smallest one. The NPPD region of s_i ($NPPD_S(s_i)$) is computed when $NPPD_S(s_0)$, $NPPD_S(s_1)$, ..., $NPPD_S(s_{i-1})$ have been already processed according to Theorem 2.1. In what follows an algorithm for drawing NPPD in the plane is presented which takes optimal time $\theta(n \log n)$.

Algorithm 1

Input: A set $S = \{s_0, s_1, ..., s_{n-1})$ of n point sites and a point p as pole in E^2.

Output: NPPD(S, p).

Begin

Step 1: Calculate the tangent of all lines $s_i p$ and make new set TS.

Step 2: Sort TS by decreasing order obtaining $TS = \{ts_0, ts_1, ..., ts_{n-1})$

Step 3: Let be $TS' = \{ts_0)\}$

Step 4: $TS = TS - \{ts_0)$

Step 5: While {$TS \neq \emptyset$;}Do

(a): Let ts_i be the maximum value of TS

(b): Do $TS' = TS' \cup \{ts_i)$ and $TS = TS - \{ts_i)$

(c): Construct $NPPD_S(s_i)$ according to theorem 1.

(d): Discard all edges inside $NPPD_S(s_i)$

Endwhile

End.

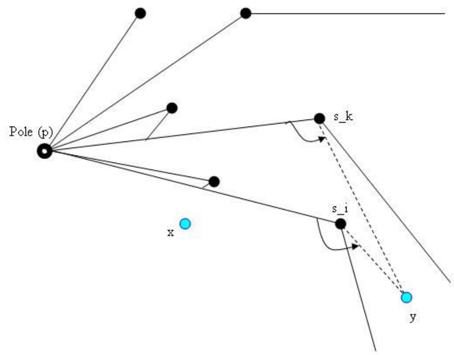

Figure 6: Incremental approach to drawing NPPD

We can solve a sorting operation in $O(n \log n)$ time and theorem 2.1 ensures this time complexity after computing all near pole polar regions.

Assume that n numbers $x_i; i = 1, ..., n$ are given. We can find a function to map the numbers into interval [−1, 1] and calculate n new numbers $x'_i; i = 1, ..., n$. Lets locate n point sites in the plane on $(1, x_i); i = 1, ..., n$ coordination and the pole p on the (0, 0) (Figure 7). Now using NPPD for these point sites with respect to the pole p we can sort the points $(1, x_i); i = 1, ..., n$. So in this way we can sort n given numbers at such time and this is a contradiction. Therefore this time complexity is a lower bound and we have the following:

Theorem 2.2. The near pole polar diagram of a set of n points in the plane with respect to a given pole p can be computed in $\theta(n \log n)$.

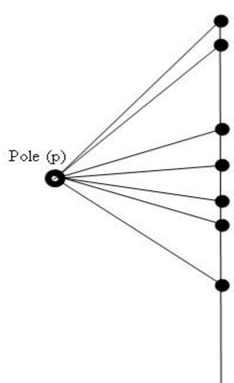

Figure 7: Contradiction: sorting n given numbers at a time less than O(n log n).

2.3 *Some applications*

In this subsection we are going to briefly address a number of applications for NPPD. But before that, it is worth mentioning that as in [3], it is possible to define and solve the contracted dual of NPPD. It can also be defined for other objects in the plane such as for line segments, convex polygons and circles.

Let S be a set of n disjoint line segments in the plane, and let p be a point not on any of the line segments of S. Using the NPPD of the line segments with respect to pole p and the Contracted dual of it, we can find all line segments of S that p can see, that is, all line segments of S that contain some point q so that the open segment pq does not intersect any line segment of S.

The pole in NPPD can be use as a light source and then we can use that in computer graphics and visibility. In addition to the applications of NPPD to computer graphics, visibility and path planning problems, it is also possible to draw decorative patterns by assigning certain points in the plane and drawing NPPD in two directions (with point sites lying on the left-hand side or the right-hand side of the pole), with application in architecture. A sample of such patterns is shown in Figures 8, 9 in which the sites lie on some concentric circles.

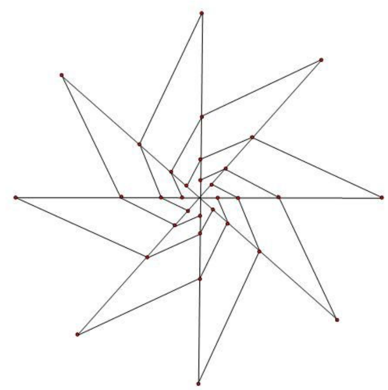

Figure 8: NPPD for some sites which lie on some concentric circles.

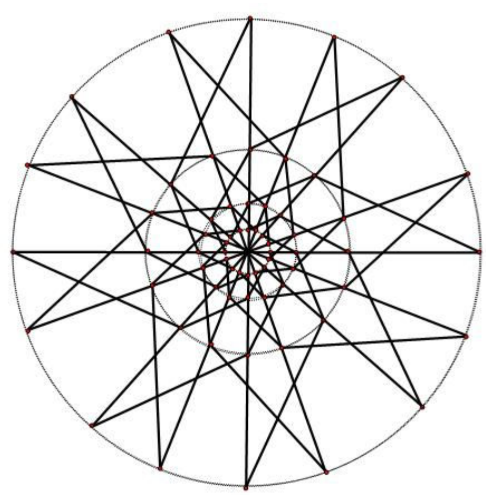

Figure 9: NPPD (in two directions) for the sites in Figure 8.

3. CONTRACTED DUAL OF NPPD

NPPD of a set of points has some useful applications that we introduced some of them in the previous section. Also the contracted dual of NPPD has some properties that makes NPPD more useful. In this section we are going to define the contracted dual of NPPD and present an algorithm (Algorithm 2) to find it. One may use this algorithm to find the contracted dual of NPPD for other objects on the plane or space. In the following we present an optimal algorithm that find the contracted dual of NPPD in $\theta(n)$ time.

We consider, two sites to be joined by edge e^* in the dual of the NPPD if and only if their corresponding faces are separated by edge e in the NPPD. So we may have some parallel edges or loops in the dual of the NPPD. If we omit the loops and replace the parallel edges with one edge, then we will have another graph named the contracted dual of the NPPD (Figure 10). In the next subsection, we present an algorithm to find the contracted dual of NPPD for a set of points.

3.1 Algorithm

Assume that n sorted point sites (with respect to y coordinate) on the plane are given and the NPPD are calculated. Here we want to present an algorithm to draw the contracted dual of NPPD and discuss the validity and complexity of the algorithm.

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Algorithm 2
Input: n sites on a plane and the NPPD of the sites.
Output: Contracted Dual of NPPD.
Begin
Step 1: Sort the sites with respect to the tangent of the lines s_i p
Step 2: Draw a straight line between each two consequent sites s_i and s_{i+1} for i = 1 to n-1.
Step 3: Insert s[1] and s[2] into an empty stack T.
Step 4: for i = 3 to n repeat
Step 4.1: while the stack is not empty and the right angle between the two above sites in stack and s[i],
(T[top - 1]T[top]s[i]) is less than π (i.e. T[top] is at the left side of the line s[i]T[top-1] ) repeat
Step 4.1.1: draw a straight line between s[i]T[top-1].
Step 4.1.2: remove the top of the stack T[top]
Step 4.2: insert s[i] in the stack
End.
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Theorem 3.1. For n given points on the plane and NPPD of them, the Contracted dual of NPPD can be found in $\theta(n)$ time using Algorithm 1.

Proof: The regions of each two consequent sites at the pole p are adjacent and then all lines drawn in step 1 belong to the CDPD. In Step 4.1, when t_i is at the left of the line $t_{i-1}\ t_{i+1}$, (the right angle $t_{i+1}\ t_i\ t_{i-1}$ is less than π) then the site t_{i+1} as a given point on the plane, is in the region of t_{i-1}. So the regions of t_{i-1} and t_{i+1} are adjacent and these points are joined to each other in step 4.1.

The regions of every two consequent sites at the pole p are adjacent and then all lines drawn in step 1 belong to the CDPD. In Step 4.1, when t_i is at the left of the line $t_{i-1}\ t_{i+1}$, (the right angle $t_{i+1}\ t_i\ t_{i-1}$ is less than π) then the site t_{i+1} as a given point on the plane, is in the region of t_{i-1}. So the regions of t_{i-1} and t_{i+1} are adjacent and these points are joined to each other in step 4.1. Now we shall show that the algorithm draws all the lines in the Contracted dual of NPPD. Assume that two sites s_i and s_j are not consequent. So there exists at least one site between them, called s_k. s_k may be at the left or right of the line $s_i\ s_j$. If it is at the right, then the half line drawn from s_k to the left and the region of s_k separate the regions of s_i and s_j. And so s_i and s_j can not connect to each other in the contracted dual of NPPD (see Figure 11.a). if s_k is at the left of the line $s_i\ s_j$, then the angle $s_i\ s_k\ s_j$ will be less than π and s_i and s_j connect to each other using the algorithm, as seen in Figure 11.b. Therefore the algorithm makes the valid solution.

Using NPPD of given sites and the data structure (tree) that we use for it, and sweeping the plane by a ray turning around the pole p, we can find the straight lines between the pole and each site in clockwise and sort the tangent of the lines $s_i\ p$ in $\theta(n)$. Step 4 has the main time usage and needs to $\theta(n)$ time. In each step, one site is added or is omitted from the stack and no site is added twice. □

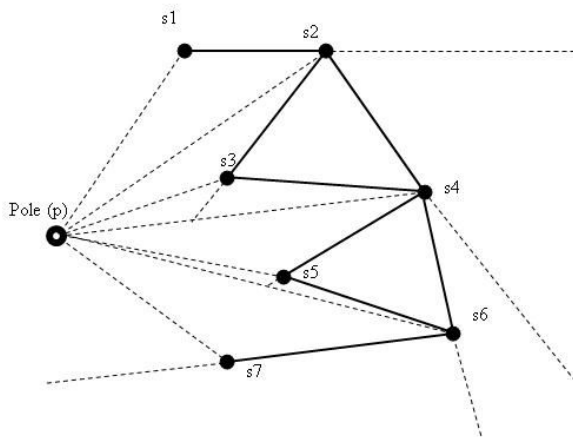

Figure 10: Contracted dual of NPPD of figure 4.

4. Conclusion

In this paper we defined the near pole polar diagram (NPPD) and the contracted dual of it for a set of point set in the plane. Then we surveyed some properties of the NPPD and use it for some applications. Also we presented two optimal algorithms for finding the NPPD of points and its contracted dual and discussed the complexity.

Calculating the NPPD of the objects and its dual are as new open problems with several applications to some visibility problems. Also it is possible to define the polar diagram in other ways and use other criteria to plane division and one can introduce some other applications of it.

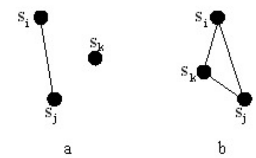

Figure 11: s_k may be at the left (b) or right (a) of the line $s_i\ s_j$.

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